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The Lausch group simplified: A direct product decomposition

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ABSTRACT

The Lausch group of a Lockett class of groups is decomposed into a direct product of cyclic groups under very general hypotheses applicable to many Fitting pairs defined in the literature. The Fitting pairs are shown to exhibit strong independence properties. The results apply to the class of all solvable groups, and, therefore, to all Lockett classes satisfying the Lockett conjecture.

1. Introduction

Various sources [1–4,6,8–11] have defined Fitting pairs (B^H, f^H) that depend upon groups $H \in \mathcal{S}$ from a particular subclass $\mathcal{S}$ of a Lockett class $\mathfrak{X} = \mathfrak{X}^*$. This paper shows that for suitable hypotheses, these pairs decompose the Lausch group as a direct product. In particular, taking the class of all solvable groups $\mathfrak{X} = \mathfrak{S}$ and using the groups $\mathcal{S}$ and Fitting pairs (B^H, f^H) defined in [6] (or [2]):

Theorem 1.1. *If $\mathcal{S}$ is the class of active groups [6] for the class of solvable groups, then the Lausch group L is isomorphic to the direct product of the groups $H/ker(f_H^H)$ for $(H) \in \mathcal{S}$.*

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Since $H/ker(f_H^H)$ is a finite abelian group, L is a product of cyclic groups. Therefore, this last theorem decomposes the Lausch group explicitly as a direct product of cyclic groups. Using a subclass of these same pairs, similar results follow for any Lockett class $\mathfrak{X}$ satisfying the Lockett conjecture ($\mathfrak{X}_* = \mathfrak{X}^* \cap \mathfrak{S}_*$).

In this form, the theorem is a special case since Hypothesis 3.1 applies to a general Lockett class $\mathfrak{X}$ and a subclass $\mathcal{S}$. Sometimes, as is the case in Theorem 1.1 above, the class $\mathcal{S}$ is sufficiently rich that it actually covers the Lausch group of $\mathfrak{X}$. That is, with appropriate hypotheses, the Fitting pairs span the Lausch group even when $\mathfrak{X} \neq \mathfrak{X}^*$. The more general theorem is:

Theorem 1.2. *If the pairs satisfy Hypothesis 3.1 for the class $\mathfrak{X}^*$ and determine the Fitting class $\mathfrak{X}_*$, then the Lausch group of $\mathfrak{X}^*$ is isomorphic to the direct product A of the groups $H/ker(f_H^H)$ for $(H) \in \mathcal{S}$. Further, the Lausch group of $\mathfrak{X}$ is isomorphic to a subgroup of A.*

This theorem will hold if the Lausch group is generated by the images in the Lausch group of the groups $H \in \mathcal{S}$. (That is, a set S of representatives H is chosen from the isomorphism classes $(H) \in \mathcal{S}$. As is usual, this relationship is made implicit by writing $H \in \mathcal{S}$.)

If conditions on the subset $\mathcal{S}$ are precise (see Hypothesis 3.1), then the groups $H \in \mathcal{S}$ will not only generate the Lausch group, but will also be independent.

Theorem 1.3. *If Hypothesis 3.1 holds and $\mathfrak{Y}$ is the Fitting class generated by all $(H) \in \mathcal{S} \setminus (K)$ for a fixed $(K) \in \mathcal{S}$ then $K \notin \mathfrak{Y}$.*

In turn, there is a set of Fitting pairs that are essentially projections.

Theorem 1.4. *If Hypothesis 3.1 holds then for each $(H) \in \mathcal{S}$ there is a Fitting pair (B^H, g^H) so that $g_H^H = f_H^H$ and $g_K^H(K) = \{1\}$ for $(K) \in \mathcal{S} \setminus (H)$.*

Of course, these theorems have hypotheses that depend upon the particular Fitting pairs and their associated Lockett class. The hypotheses needed for these theorems are very general and apply to a wide class of naturally defined Fitting pairs. That is, the preceding three theorems are very general. Using the Fitting pairs defined in [2] or [8] these results extend beyond $\mathfrak{S}$ and classes satisfying the Lockett conjecture.

The results in this paper require very little from the theory of Fitting classes. They depend mainly on simple arguments about linear independence.

2. Background

Either [6] or [7] provides enough background for this paper.

A Fitting class $\mathfrak{X}$ in the class $\mathfrak{S}$ of all solvable groups is **normal** if an $\mathfrak{X}$-injector of a group G in $\mathfrak{S}$ is a normal subgroup of G. Blessenohl and Gaschütz [3] showed for normal