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A supercharacter theory for involutive algebra groups ☆

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If $\mathfrak{J}$ is a finite-dimensional nilpotent algebra over a finite field $\Bbbk$, the algebra group $P = 1 + \mathfrak{J}$ admits a (standard) supercharacter theory as defined in [16]. If $\mathfrak{J}$ is endowed with an involution σ, then σ naturally defines a group automorphism of $P = 1 + \mathfrak{J}$, and we may consider the fixed point subgroup $C_P(\sigma) = \{x \in P : \sigma(x) = x^{-1}\}$. Assuming that $\Bbbk$ has odd characteristic p, we use the standard supercharacter theory for P to construct a supercharacter theory for $C_P(\sigma)$. In particular, we obtain a supercharacter theory for the Sylow p-subgroups of the finite classical groups of Lie type, and thus extend in a uniform way the construction given by André and Neto in [7,8] for the special case of the symplectic and orthogonal groups.

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1. Introduction

The notion of a supercharacter theory of a finite group was introduced by P. Diaconis and I.M. Isaacs in [16] to generalise the *basic characters* defined by C. André in [2–4], and the *transition characters* defined by N. Yan in his PhD thesis [23] (see also [24]). Both basic and transition characters were introduced with the aim of approaching the usual character theory of the finite group $\mathrm{UT}_n(\Bbbk)$ consisting of $n \times n$ unimodular upper-triangular matrices over a finite field $\Bbbk$ of characteristic p. (By "unimodular", we mean that all diagonal entries are equal to 1; we will refer to $\mathrm{UT}_n(\Bbbk)$ simply as a (finite) *unitriangular group*.) The basic idea is to coarsen the usual character theory of a group by replacing irreducible characters with linear combinations of irreducible characters that are constant on a set of clumped conjugacy classes.

Let G be a finite group, and write $\mathrm{Irr}(G)$ to denote the set of irreducible characters of G. (Throughout the paper, all characters are taken over the field $\mathbb{C}$ of complex numbers.) Let $\mathcal{K}$ be a partition of G, and let $\mathcal{X}$ be a partition of $\mathrm{Irr}(G)$. (Here, and throughout this paper, when we use the word "partition", we require that the parts are all non-empty.) For each $X \in \mathcal{X}$, we define

$$\sigma_X = \sum_{\psi \in X} \psi(1)\psi, \tag{1a}$$

and note that $\sum_{X \in \mathcal{X}} \sigma_X = \rho_G$, the regular character of G. (Recall that $\rho_G(g) = 0$ for all $g \in G$, $g \neq 1$, and $\rho_G(1) = |G|$.) We recall from [16] that the pair $(\mathcal{X}, \mathcal{K})$ is called a *supercharacter theory* for G provided that the following conditions hold.

(S1) $|\mathcal{X}| = |\mathcal{K}|$.
(S2) $\{1\} \in \mathcal{K}$.
(S3) For each $X \in \mathcal{X}$, the character σ_X is constant on each member of $\mathcal{K}$.

As shown in [16, Lemma 2.1] this definition is equivalent to the following (see [9]). A *supercharacter theory* for a finite group G is a pair $(\mathcal{X}, \mathcal{K})$ where $\mathcal{K}$ is a partition of G, $\mathcal{X}$ is a collection of characters of G, and the following conditions hold.

(S1′) $|\mathcal{X}| = |\mathcal{K}|$.
(S2′) Every irreducible character of G is a constituent of a unique $\chi \in \mathcal{X}$.
(S3′) Every $\chi \in \mathcal{X}$ is constant on each member of $\mathcal{K}$.

We refer to the elements of $\mathcal{X}$ as the *supercharacters* of G, and to each $K \in \mathcal{K}$ as a *superclass* of G. Regardless of which definition one chooses to work with, it is straightforward to verify that each superclass is a union of conjugacy classes of G and that each of the partitions $\mathcal{K}$ and $\mathcal{X}$ determines the other. The only significant difference between these two definitions is that the second approach can yield supercharacters which are multiples of the characters σ_X defined above.