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A classification of primitive permutation groups with finite stabilizers

Simon M. Smith

Department of Mathematics, NYC College of Technology, City University of New York (CUNY), New York, NY, USA

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ABSTRACT

We classify all infinite primitive permutation groups possessing a finite point stabilizer, thus extending the seminal Aschbacher–O'Nan–Scott Theorem to all primitive permutation groups with finite point stabilizers.

1. Introduction

Recall that a transitive permutation group G on a set Ω is primitive if the only G-invariant partitions are $\{\{\alpha\} : \alpha \in \Omega\}$ and $\{\Omega\}$. In the finite case, these groups are the fundamental actions from which all permutation groups are constituted.

The finite primitive permutation groups were classified by the famous Aschbacher–O'Nan–Scott Theorem, first stated independently by O'Nan and Scott. Scott's initial

E-mail address: sismith@citytech.cuny.edu.

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statement [15] omitted the class of twisted wreath products; an extended and corrected version of the theorem appears in [1] and [7]. The Aschbacher–O'Nan–Scott Theorem describes in detail the structure of finite primitive permutation groups in terms of finite simple groups. A modern statement of the theorem with a self-contained proof can be found in [8].

Primitive permutation groups with finite point stabilizers are precisely those primitive groups whose subdegrees are bounded above by a finite cardinal [14,2]. This class of groups also includes all infinite primitive permutation groups that act regularly on some finite self-paired suborbit (see [11, Problem 7.51]). Our main result is Theorem 1.1 below; in conjunction with the Aschbacher–O'Nan–Scott Theorem, this yields a satisfying classification of all primitive permutation groups with finite point stabilizers, describing in detail their structure in terms of finitely generated simple groups.

Theorem 1.1. *If $G \leq \mathrm{Sym}(\Omega)$ is an infinite primitive permutation group with a finite point stabilizer G_α, then G is finitely generated by elements of finite order and possesses a unique (non-trivial) minimal normal subgroup M; there exists an infinite, nonabelian, finitely generated simple group K such that $M = K_1 \times \cdots \times K_m$, where $m \geq 1$ is finite and $K_i \cong K$ for $1 \leq i \leq m$; the stabilizer G_α acts transitively on the components $K_1, \ldots, K_m$ of M by conjugation; and G falls into precisely one of the following categories:*

(i) *M is simple and acts regularly on Ω, and G is equal to the split extension $M.G_\alpha$ for some $\alpha \in \Omega$, with no non-identity element of G_α inducing an inner automorphism of M;*

(ii) *M is simple, and acts non-regularly on Ω, with M of finite index in G and $M \leq G \leq \mathrm{Aut}(M)$;*

(iii) *M is non-simple. In this case $m > 1$, and G is permutation isomorphic to a subgroup of the wreath product $H \,\mathrm{Wr}_\Delta\, \mathrm{Sym}(\Delta)$ acting via the product action on Γ^m, where $\Delta = \{1, \ldots, m\}$, Γ is some infinite set and $H \leq \mathrm{Sym}(\Gamma)$ is an infinite primitive group with a finite point stabilizer. Here K is the unique minimal normal subgroup of H. Moreover, if M is regular, then H is of type* (i) *and if M is non-regular then H is of type* (ii).

For each type (i), (ii) and (iii) there exist examples of infinite primitive permutation groups with finite point stabilizers. We present these in Section 4.

For permutation groups which lie in classes (i) and (iii) there are known conditions which guarantee primitivity (Proposition 3.3 and the paragraph immediately following it, and Lemma 3.6).

For any group G of type (iii), an explicit permutation embedding of G into $H \,\mathrm{Wr}\, \mathrm{Sym}(\Delta)$ is described in Lemma 3.1 and its proof.

This paper is not the first to extend the Aschbacher–O'Nan–Scott Theorem to specific classes of infinite groups. In [9], a version of the Aschbacher–O'Nan–Scott Theorem for