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“Strong” Euler class of a stably free module of odd rank

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ABSTRACT

Let R be a commutative Noetherian ring of dimension $n \geq 3$. Following a suggestion of Fasel, we establish a group homomorphism ϕ from van der Kallen's group $\mathrm{Um}_{n+1}(R)/E_{n+1}(R)$ to the n-th Euler class group $E^n(R)$ so that: (1) when n is even, ϕ coincides with the homomorphism given by Bhatwadekar and Sridharan through Euler classes; (2) when n is odd, ϕ is non-trivial in general for an important class of rings; (3) the sequence $Um_{n+1}(R)/E_{n+1}(R) \xrightarrow{\phi} E^n(R) \longrightarrow E_0^n(R) \to 0$ is exact, where $E_0^n(R)$ is the n-th weak Euler class group. (If $X = \mathrm{Spec}(R)$ is a smooth affine variety of dimension n over $\mathbb{R}$ so that the complex points of X are complete intersections and the canonical module K_R is trivial, then the sequence is proved to be exact on the left as well.) More generally, let R be a commutative Noetherian ring of dimension d and n be an integer such that $n \leq d \leq 2n-3$. We also indicate how to extend our arguments to this setup to obtain a group homomorphism from $\mathrm{Um}_{n+1}(R)/E_{n+1}(R)$ to $E^n(R)$.

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1. Introduction

This paper may be regarded as a supplementary note to the articles [4,5,11,15]. Let R be a Noetherian ring of (Krull) dimension n. Our purpose is to understand the relation between the following groups:

(1) $\underline{Um_{n+1}(R)/E_{n+1}(R)}$: the orbit space of unimodular rows of length $n+1$ under the natural action of elementary $(n+1) \times (n+1)$ matrices. This space is equipped with a group structure, introduced by van der Kallen [14].

(2) $\underline{E^n(R)}$: the n-th Euler class group of R defined by Bhatwadekar and Sridharan [2,4] which detects the obstruction for a projective R-module of rank n (with trivial determinant) to split off a free summand of rank one.

(3) $\underline{E_0^n(R)}$: the n-th weak Euler class group of R defined by Bhatwadekar and Sridharan [3,4] which is a certain quotient of $E^n(R)$ and is an analogue of the Chow group $CH_0(R)$ for regular R.

When n is *even* and $\mathbb{Q} \subset R$, Bhatwadekar and Sridharan established a wonderful relation between these groups in [4, 7.6] by showing that there is an exact sequence:

$$(*) \quad \mathrm{Um}_{n+1}(R)/E_{n+1}(R) \longrightarrow E^n(R) \longrightarrow E_0^n(R) \longrightarrow 0$$

Let $[a_1, \cdots, a_{n+1}] \in Um_{n+1}(R)/E_{n+1}(R)$. The first map in the above exact sequence is given by the Euler class of the stably free R-module P associated to the unimodular row $[a_1, \cdots, a_{n+1}]$ (we shall freely use the same notation for a unimodular row and the elementary orbit represented by it). We may loosely call it the Euler class of $[a_1, \cdots, a_{n+1}]$ and denote it by $e[a_1, \cdots, a_{n+1}]$. Thus, $e[a_1, \cdots, a_{n+1}] = 0$ if and only if $P \simeq Q \oplus R$ for some R-module Q, and equivalently, $[a_1, \cdots, a_{n+1}]$ is the first row of a right-invertible $2 \times (n+1)$ matrix. If $X = \mathrm{Spec}(R)$ is a smooth affine variety of dimension n (n even) over $\mathbb{R}$ such that $X(\mathbb{R})$, the smooth real manifold consisting of all real points of X, is orientable and every complex maximal ideal of R is a complete intersection, a remarkable result of Fasel [11, 5.9] essentially asserts that $e[a_1, \cdots, a_{n+1}] = 0$ if and only if $[a_1, \cdots, a_{n+1}]$ is the first row of an $(n+1) \times (n+1)$ elementary matrix.

All the results mentioned above are heavily dependent on the fact that n is even. Whereas, if n is odd, the Euler class $e[a_1, \cdots, a_{n+1}]$ is always trivial (note that if n is odd, $[a_1, \cdots, a_{n+1}]$ is the first row of a right-invertible $2 \times (n+1)$ matrix). It is therefore natural to ask whether for odd n one can define a morphism $\phi : Um_{n+1}(R)/E_{n+1}(R) \longrightarrow E^n(R)$ which is non-trivial for some important class of rings and further, we have an exact sequence as above. In this article, we answer this question affirmatively in 3.4, 3.6, 3.8, when R is a commutative Noetherian ring (unlike [4] we do not assume that $\mathbb{Q} \subset R$). We call the element $\phi[a_1, \cdots, a_{n+1}] \in E^n(R)$ the *"strong" Euler class* of $[a_1, \cdots, a_{n+1}]$. The definition of ϕ is for general n and it coincides with $e[a_1, \cdots, a_{n+1}]$ when n is even and $\mathbb{Q} \subset R$ (see 3.10).