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Cell algebras

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ABSTRACT

A family of associative algebras called cell algebras is defined and studied. These algebras generalize the cellular algebras of Graham and Lehrer. Standard results for cellular algebras carry over nicely to the more general cell algebras, including the characterization of their irreducible modules in terms of a bilinear form, a description of their decomposition and Cartan matrices, and a description of their hereditary ideals and possible quasi-hereditary algebra structures.

As examples of cell algebras which are not cellular, the semigroup algebras $R[\mathcal{T}_r]$ and $R[\mathcal{PT}_r]$ corresponding to the full transformation semigroup $\mathcal{T}_r$ and the partial transformation semigroup $\mathcal{PT}_r$ are shown to be cell algebras and cell algebra bases are obtained for these (and related) algebras. The general cell algebra theory is then applied to classify the irreducible representations of these algebras (when R is any field of characteristic 0 or p). In certain cases the algebras are found to be quasi-hereditary.

1. Introduction

Cellular algebras were introduced by Graham and Lehrer in [6], unifying the representation theory of a large number of algebras including Hecke algebras, Schur algebras,

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q-Schur algebras, Brauer algebras, Temperley–Lieb algebras, and others. More recently, East, Wilcox, Guo and Xi, and others have studied the cellularity of certain semigroup algebras [4,14] and twisted semigroup algebras [14,7]. However, the definition of a cellular algebra requires that the algebra carry an anti-automorphism $*$ with $x^{**} = x$ and this limits the applicability of the cellular theory.

In this paper we define and study a new class of associative algebras, called cell algebras, which includes the cellular algebras as a special case. These cell algebras have bases with nice multiplicative properties similar to those for cellular algebras, but need not have anti-automorphisms. We expect that many cellular algebras will have natural generalizations to cell algebras. For example, we will show that the semigroup algebras $R[M]$ are cell algebras when $M = \mathcal{T}_r$, the full transformation semigroup, or $M = \mathcal{PT}_r$, the partial transformation semigroup. Standard results for cellular algebras carry over nicely to the more general cell algebras. Our presentation follows closely the development of cellular algebras as set forth in [6] or [12].

In Sections 2 through 6 we develop the general theory of cell algebras including their irreducible modules (Section 4), their decomposition and Cartan matrices (Section 5) and hereditary ideals (Section 6). In Sections 7 through 9 we apply the theory to the semigroup algebras $R[M]$ for M any submonoid of $\mathcal{PT}_r$ containing the symmetric group $\mathfrak{S}_r$. In particular, these include the cases when M is $\mathcal{PT}_r$, $\mathcal{T}_r$, or the Rook monoid (symmetric inverse semigroup) $\mathfrak{R}_r$. In Section 10 we consider briefly other possible applications of the cell algebra theory.

2. Cell bases and cell modules

Let R be a commutative integral domain with unit 1 and let A be an associative, unital R-algebra. Let Λ be a finite set with a partial order $\leqslant$ and for each $\lambda \in \Lambda$ let $L(\lambda), R(\lambda)$ be finite sets of "left indices" and "right indices". Assume that for each $\lambda \in \Lambda, s \in L(\lambda)$, and $t \in R(\lambda)$ there is an element ${}_sC_t^\lambda \in A$ such that the map $(\lambda, s, t) \mapsto {}_sC_t^\lambda$ is injective and

$$C = \left\{{}_sC_t^\lambda : \lambda \in \Lambda,\ s \in L(\lambda),\ t \in R(\lambda)\right\}$$

is a free R-basis for A. Define R-submodules of A by

$$A^\lambda = R\text{-span of } \left\{{}_sC_t^\mu : \mu \in \Lambda,\ \mu \geqslant \lambda,\ s \in L(\mu),\ t \in R(\mu)\right\}$$

and

$$\hat{A}^\lambda = R\text{-span of } \left\{{}_sC_t^\mu : \mu \in \Lambda,\ \mu > \lambda,\ s \in L(\mu),\ t \in R(\mu)\right\}.$$

Definition 2.1. Given A, Λ, C, A is a cell algebra with poset Λ and cell basis C if

i For any $a \in A$, $\lambda \in \Lambda$, and $s, s' \in L(\lambda)$, there exists $r_L = r_L(a, \lambda, s, s') \in R$ such that, for any $t \in R(\lambda)$, $a \cdot {}_sC_t^\lambda = \sum_{s' \in L(\lambda)} r_L \cdot {}_{s'}C_t^\lambda \bmod \hat{A}^\lambda$, and