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Lengths and multiplicities of integrally closed modules over a two-dimensional regular local ring

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ABSTRACT

Let $(R, \mathbf{m})$ be a two-dimensional regular local ring with infinite residue field. We prove an analogue of the Hoskin–Deligne length formula for a finitely generated, torsion-free, integrally closed R-module M. As a consequence, we get a formula for the Buchsbaum–Rim multiplicity of F/M, where $F = M^{**}$.

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1. Introduction

The theory of integrally closed or complete ideals in a two-dimensional regular local ring was founded by Zariski in [14]. Since, then this theory has received a good deal of attention and has been refined and generalised. The first named author generalised this theory to finitely generated, torsion-free, integrally closed modules in [9,8]. While the structural cornerstones of this theory are Zariski's product and unique factorisation theorems, the basic numerical result here is the Hoskin–Deligne length formula.

This formula has several proofs beginning with the one by Hoskin in [5], through proofs by Deligne in [3], by Rees in [12], by Lipman in [10], by Johnston and Verma in [7], and by the first named author in [9] (which is based on techniques of [10] and [6]), to the one in [2].

In this paper we obtain an analogue of the Hoskin–Deligne formula for finitely generated, torsion-free, integrally closed modules over a two-dimensional regular local ring. A consequence of this is a formula for the Buchsbaum–Rim multiplicity of a certain finite length module associated to an integrally closed module.

We now summarise the rest of the paper. In Section 2, we collect various facts and results about integrally closed modules and reductions from [13], about integrally closed modules and their transforms over two-dimensional regular local rings from [8] and about Buchsbaum–Rim multiplicities from [1]. In Section 3, we prove the analogue of the Hoskin–Deligne formula for integrally closed modules which expresses their colength in terms of those of modules contracted from the order valuation rings of various quadratic transforms of the base ring. In the final Section 4 we apply our analogue of the Hoskin–Deligne formula to prove a Buchsbaum–Rim multiplicity formula for such modules.

2. Preliminaries

2.1. Integral closures and reductions of modules

We review the notions of integral closures and reductions for torsion-free modules over arbitrary Noetherian domains as developed by Rees [13].

Throughout this subsection, R will be a Noetherian domain with field of fractions K and M will be a finitely generated, torsion-free R-module. We denote its rank by $\mathrm{rk}_R(M)$. By M_K we denote the $\mathrm{rk}_R(M)$-dimensional K-vector space $M \otimes_R K$. If N is a submodule of M then N_K is naturally identified with a subspace of M_K.

Any ring between R and K is said to be a birational overring of R. For any such birational overring S of R, we let MS denote the S-submodule of M_K generated by M. There is a canonical R-module homomorphism from $M \otimes_R S$ onto MS with kernel being the submodule of S-torsion (equivalently R-torsion) elements. Hence, $M \otimes_R S$ modulo S-torsion and MS are isomorphic as S-modules.

Let $S(M)$ denote the image of the symmetric algebra $\mathrm{Sym}^R(M)$ in the algebra $\mathrm{Sym}^K(M_K)$ under the canonical map. As an R-algebra $S(M)$ is $\mathrm{Sym}^R(M)$ modulo