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Orthogonal multiple flag varieties of finite type I: Odd degree case ☆

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A R T I C L E I N F O

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Let G be the split orthogonal group of degree $2n+1$ over an arbitrary infinite field $\mathbb{F}$ of char $\mathbb{F} \neq 2$. In this paper, we classify multiple flag varieties $G/P_1 \times \cdots \times G/P_k$ of finite type. Here a multiple flag variety is said to be of finite type if it has a finite number of G-orbits with respect to the diagonal action of G.

1. Introduction

In [4], Magyar, Weyman and Zelevinsky classified multiple flag varieties for $\mathrm{GL}_n(\mathbb{F})$ of finite type. In their subsequent paper [5], they classified multiple flag varieties for $\mathrm{Sp}_{2n}(\mathbb{F})$ of finite type. (They assume $\mathbb{F}$ is algebraically closed.)

Recently the author gave explicit orbit decomposition for an example of orthogonal case in [6]. In this paper, we will classify multiple flag varieties of finite type for the split orthogonal group of degree $2n+1$.

Let $\mathbb{F}$ be an arbitrary commutative infinite field of char $\mathbb{F} \neq 2$. Let (,) denote the symmetric bilinear form on $\mathbb{F}^{2n+1}$ defined by

$$(e_i, e_j) = \delta_{i,2n+2-j}$$

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for $i, j = 1, \ldots, 2n+1$ where $e_1, \ldots, e_{2n+1}$ is the canonical basis of $\mathbb{F}^{2n+1}$. Define the split orthogonal group

$$G = \left\{ g \in \mathrm{GL}_{2n+1}(\mathbb{F}) \;\middle|\; (gu, gv) = (u, v) \text{ for all } u, v \in \mathbb{F}^{2n+1} \right\}$$

with respect to this form. Let us write $G = \mathrm{O}_{2n+1}(\mathbb{F})$ in this paper.

A subspace V of $\mathbb{F}^{2n+1}$ is called isotropic if $(V, V) = \{0\}$. For a sequence $\mathbf{a} = (\alpha_1, \ldots, \alpha_p)$ of positive integers such that $\alpha_1 + \cdots + \alpha_p \le n$, there corresponds the flag variety

$$M_{\mathbf{a}} = \left\{ V_1 \subset \cdots \subset V_p \;\middle|\; \dim V_i = \alpha_1 + \cdots + \alpha_i \text{ for } i = 1, \ldots, p,\ (V_p, V_p) = \{0\} \right\}.$$

For the canonical flag

$$\mathcal{F}_0 : \mathbb{F}e_1 \oplus \cdots \oplus \mathbb{F}e_{\alpha_1} \subset \cdots \subset \mathbb{F}e_1 \oplus \cdots \oplus \mathbb{F}e_{\alpha_1 + \cdots + \alpha_p}$$

in $M_{\mathbf{a}}$, the isotropic subgroup for $\mathcal{F}_0$ in G is a standard parabolic subgroup $P_{\mathbf{a}}$ consisting of elements in G of the form

$$\begin{pmatrix} A_1 & & & & & & * \\ & \ddots & & & & & \\ & & A_p & & & & \\ & & & A_0 & & & \\ & & & & A_p^* & & \\ & & & & & \ddots & \\ 0 & & & & & & A_1^* \end{pmatrix}$$

with $A_i \in \mathrm{GL}_{\alpha_i}(\mathbb{F})$ for $i = 1, \ldots, p$ and $A_0 \in \mathrm{O}_{2\alpha_0+1}(\mathbb{F})$ ($\alpha_0 = n - (\alpha_1 + \cdots + \alpha_p)$) where $A_i^* = J_{\alpha_i} {}^t A_i^{-1} J_{\alpha_i}$ for $i = 1, \ldots, p$ and J_m is the $m \times m$ matrix given by

$$J_m = \begin{pmatrix} 0 & & 1 \\ & \cdot^{\cdot^{\cdot}} & \\ 1 & & 0 \end{pmatrix}.$$

Since $M_{\mathbf{a}}$ is G-homogeneous, we can identify $M_{\mathbf{a}}$ with $G/P_{\mathbf{a}}$.

Remark 1.1. Define the split special orthogonal group

$$G_0 = \{ g \in G \mid \det g = 1 \} \quad \bigl(= \mathrm{SO}_{2n+1}(\mathbb{F})\bigr).$$

Then $G = G_0 \sqcup (-I_{2n+1})G_0$. Since $-I_{2n+1}$ acts trivially on $M_{\mathbf{a}}$, G_0-orbits on $M_{\mathbf{a}}$ are the same as G-orbits.