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Large subgroups of simple groups

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Let G be a finite group. A proper subgroup H of G is said to be large if the order of H satisfies the bound $|H|^3 \geqslant |G|$. In this note we determine all the large maximal subgroups of finite simple groups, and we establish an analogous result for simple algebraic groups (in this context, largeness is defined in terms of dimension). An application to triple factorisations of simple groups (both finite and algebraic) is discussed.

Dedicated to the memory of Ákos Seress

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1. Introduction

Let G be a group. A *triple factorisation* of G is a factorisation of the form $G = ABA$, where A and B are proper subgroups of G. Such factorisations arise naturally in several different contexts. For example, the Bruhat decomposition $G = BNB$ of a group of Lie type, where B is a Borel subgroup and N is the normaliser of a maximal torus, plays an important role in the study of such groups. In a different direction, a triple factorisation corresponds to a flag-transitive point-line incidence geometry in which each pair of points is incident with at least one line (see [13, Lemma 3]).

Determining the triple factorisations of a given group is a difficult problem. For a finite group G, a starting point is the easy observation that $G = ABA$ only if

$$\max\{|A|^3, |B|^3\} \geqslant |G|. \tag{1}$$

This motivates the following definition.

Definition 1. Let G be a finite group. A proper subgroup H of G is said to be *large* if the order of H satisfies the bound $|H|^3 \geqslant |G|$.

In [2], Alavi and Praeger develop a general framework for studying triple factorisations of finite groups in terms of group actions. In particular, a reduction strategy for classifying triple factorisations is presented in [2, Section 1.1], in which it is reasonable to assume that the subgroup A appearing in a factorisation $G = ABA$ is maximal (and also core-free). Simultaneous triple factorisations of the form $G = ABA = BAB$ are particularly interesting from a geometric point of view, and in this situation it is reasonable to assume that both A and B are maximal subgroups of G. Factorisations of this form have been studied in two recent papers [1,12]. In [1], $G = \mathrm{GL}(V)$ and the subgroups A and B either stabilise a subspace of V, or stabilise a decomposition $V = V_1 \oplus V_2$ with $\dim V_1 = \dim V_2$. In [12], the case where $G = S_n$ and A, B are maximal conjugate subgroups is investigated.

The main aim of this paper is to determine the large maximal subgroups of finite simple groups, which can be viewed as a first step towards a general investigation of triple factorisations of simple groups. By the Classification of Finite Simple Groups, every nonabelian finite simple group is isomorphic to an alternating group A_n of degree $n \geqslant 5$, a group of Lie type defined over a finite field $\mathbb{F}_q$ (of classical or exceptional type), or one of 26 sporadic groups. There is a vast literature on the subgroup structure of finite simple groups, and their maximal subgroups in particular.

The problem of determining the “large” maximal subgroups of finite simple groups has a long history, with many applications. For alternating groups, it is closely related to the following old question in permutation group theory (see [37] and the references therein): how large can a primitive group G of degree n be, assuming G does not contain A_n? For groups of Lie type, some related results are established in [23] and [26]. The main theorem