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On cliques in edge-regular graphs

Leonard H. Soicher

School of Mathematical Sciences, Queen Mary University of London, Mile End Road, London E1 4NS, UK

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ABSTRACT

Let Γ be an edge-regular graph with given parameters (v, k, λ). We show how to apply a certain "block intersection polynomial" in two variables to determine a good upper bound on the clique number of Γ, and to obtain further information concerning the cliques S of Γ with the property that every vertex of Γ not in S is adjacent to exactly m or $m+1$ vertices of S, for some constant $m \geq 0$. Some interesting examples are studied using computation with groups and graphs.

Dedicated to the memory of Ákos Seress

1. Introduction

In this paper we present new results concerning the cliques in an edge-regular graph Γ with given parameters. We show how to apply a certain "block intersection polynomial" [5,12] to determine a good upper bound on the clique number of Γ, and to obtain

E-mail address: L.H.Soicher@qmul.ac.uk.

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information on the cliques S of Γ with the property that every vertex of Γ not in S is adjacent to exactly m or $m+1$ vertices of S, for some constant $m \geq 0$.

Every orbital graph for a finite transitive permutation group is edge-regular, and we study some informative examples of orbital graphs using the permutation group functionality in GAP [8], to which Ákos Seress was a major contributor, together with the GAP package GRAPE [13] for computing with graphs with groups acting on them.

2. Definitions and background

All graphs in this paper are finite and undirected, with no loops and no multiple edges. A graph Γ is *edge-regular* with *parameters* (v, k, λ) if Γ has exactly v vertices, is regular of valency k, and every pair of adjacent vertices have exactly λ common neighbours. An *orbital graph* for a transitive permutation group G on a finite set Ω is a graph with vertex set Ω and edge set the G-orbit of some unordered pair $\{\alpha, \beta\}$ of distinct vertices, such that α and β are interchanged by some element of G. Such orbital graphs are edge-regular, and provide us with interesting examples. A graph Γ is *strongly regular* with *parameters* (v, k, λ, μ) if Γ is edge-regular with parameters (v, k, λ), and every pair of distinct nonadjacent vertices have exactly μ common neighbours. A *clique* in a graph Γ is a set of pairwise adjacent vertices, an *s-clique* is a clique of size s, and a *maximum clique* of Γ is a clique of the largest size in Γ. The size of a maximum clique in Γ, its *clique number*, is denoted by $\omega(\Gamma)$. The set of vertices adjacent to a vertex v in a graph Γ is denoted by $\Gamma(v)$.

For $n \geq k > 0$, the *Kneser graph* $K(n, k)$ has as vertices the k-subsets of $\{1, \dots, n\}$, with two vertices adjacent precisely when they are disjoint. For example, $K(5, 2)$ is the Petersen graph. Observe that $K(n, k)$ is edge-regular, with parameters $(\binom{n}{k}, \binom{n-k}{k}, \binom{n-2k}{k})$.

A *regular clique*, or more specifically, an *m-regular clique* in a graph Γ is a clique S such that every vertex of Γ not in S is adjacent to exactly m vertices of S, for some constant $m > 0$. For example, if $n \geq 3$, then each maximum clique of $K(2n, 2)$ is $(n-2)$-regular. A *quasiregular clique*, or more specifically, an *m-quasiregular clique* in a graph Γ is a clique S of size at least 2, such that every vertex of Γ not in S is adjacent to exactly m or $m+1$ vertices of S, for some constant $m \geq 0$. For example, if $n \geq 3$, then each maximum clique of $K(2n-1, 2)$ is $(n-3)$-quasiregular, and each maximum clique of $K(3n, 3)$ is $(n-3)$-quasiregular. Note that a clique S of size at least 2 in a graph is m-regular precisely when S is both $(m-1)$-quasiregular and m-quasiregular.

Suppose that Γ is a strongly regular graph of valency $k > 0$ and having least eigenvalue σ (the eigenvalues of (the adjacency matrix of) Γ are determined by its parameters). In his famous thesis, Delsarte [6] proved that

$$\omega(\Gamma) \leq \lfloor 1 - k/\sigma \rfloor. \tag{1}$$

Moreover, if Γ is connected and not complete, then a clique S of Γ is regular if and only if $|S| = 1 - k/\sigma$ (see [4, Proposition 1.3.2(ii)]).