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s-Arc-transitive graphs and normal subgroups [☆]

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We study s-arc-transitive graphs with $s \geqslant 2$, and give a characterisation of the actions of vertex-transitive normal subgroups. An interesting consequence of this characterisation states that each non-bipartite 3-arc-transitive graph is a normal cover of a 2-arc-transitive graph admitting a simple group, or a locally primitive graph admitting a simple group with a soluble vertex stabiliser.

1. Introduction

Denote by $\Gamma = (V, E)$ an undirected simple connected graph with vertex set V and edge set E. For a vertex $\alpha \in V$, denote by $\Gamma(\alpha)$ the set of vertices adjacent to α

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in Γ. For a positive integer s, an *s-arc* of Γ is an $(s+1)$-tuple $(\alpha_0, \alpha_1, \cdots, \alpha_s)$ of vertices such that $\alpha_i \in \Gamma(\alpha_{i-1})$ for $1 \leqslant i \leqslant s$ and $\alpha_{i-1} \neq \alpha_{i+1}$ for $1 \leqslant i \leqslant s-1$. If Γ is regular and $G \leqslant \mathsf{Aut}\,\Gamma$ is transitive on the set of s-arcs of Γ, then Γ is called a *(G, s)-arc-transitive* graph. A (G, s)-arc-transitive graph is sometimes simply called an *s-arc-transitive graph*, while a $(G, 1)$-arc-transitive is simply called *G-arc-transitive*. A (G, s)-arc-transitive graph which is not $(G, s+1)$-arc-transitive is simply called *(G, s)-transitive*.

Interest in s-arc-transitive graphs stems from a beautiful result of Tutte (1947) who proved that there exist no 6-arc-transitive trivalent graphs. Tutte's result was generalised by Weiss (1981) who proved that there exists no finite s-arc-transitive graph of valency at least 3 for $s \geqslant 8$. Since then, characterising s-arc-transitive graphs has received considerable attention in the literature (see for example [5,7,10,11,13,15]). Recall that, for a group H and a prime p, denote by $\mathbf{O}_p(H)$ the largest normal p-subgroup of H. For a group $G \leqslant \mathsf{Aut}\,\Gamma$ and a vertex α, let $G_\alpha^{[1]}$ be the kernel of G_α acting on $\Gamma(\alpha)$, and $G_\alpha^{\Gamma(\alpha)} \cong G_\alpha/G_\alpha^{[1]}$ be the induced permutation group of G_α on $\Gamma(\alpha)$. A G-arc-transitive graph Γ is called *G-locally primitive* if G_α acts on $\Gamma(\alpha)$ primitively, that is, the induced permutation group $G_\alpha^{\Gamma(\alpha)}$ is primitive. A G-arc-transitive graph is $(G, 2)$-arc-transitive if and only if $G_\alpha^{\Gamma(\alpha)}$ is a 2-transitive permutation group.

The first result of this paper characterises the action of a vertex-transitive normal subgroup of a group G which acts on a graph s-arc-transitively.

Theorem 1.1. *Let $\Gamma = (V, E)$ be a connected (G, s)-transitive graph where $s \geqslant 2$, and let N be a normal subgroup of G which is transitive on V. Then either N is regular on V, or at least one of the following holds, where $\{\alpha, \beta\}$ is an edge and p is a prime.*

- (i) *Γ is (N, s)-transitive;*
- (ii) *Γ is $(G, 5)$-transitive and $(N, 4)$-transitive;*
- (iii) *Γ is $(G, 3)$-transitive and $(N, 2)$-transitive;*
- (iv) *Γ is $(G, 3)$-transitive and N-locally primitive of valency p^f, $G_\alpha^{\Gamma(\alpha)} \leqslant \mathrm{A\Gamma L}_1(p^f)$, and the number of $N_{\alpha\beta}^{\Gamma(\alpha)}$-orbits on $\Gamma(\alpha) \setminus \{\beta\}$ divides $\gcd(p^f - 1, f)^2$;*
- (v) *Γ is $(G, 2)$-transitive and N-locally primitive of valency 28, $G_\alpha = (G_\alpha^{[1]} \times \mathrm{PSL}_2(8)).\mathbb{Z}_3$, and $N_\alpha = N_\alpha^{[1]} \times \mathrm{PSL}_2(8)$, where $G_\alpha^{[1]} \trianglelefteq \mathbb{Z}_9{:}\mathbb{Z}_6$ and $N_\alpha^{[1]} \trianglelefteq \mathrm{D}_{18}$;*
- (vi) *Γ is $(G, 2)$-transitive of valency p^f, $G_\alpha^{\Gamma(\alpha)}$ is affine, $G_\alpha = \mathbf{O}_p(N_\alpha){:}G_{\alpha\beta}$, $\mathbf{O}_p(N_\alpha) = \mathbf{O}_p(G_\alpha) \cong \mathbb{Z}_p^f$, and either*
  - (a) *$N_{\alpha\beta} \cong \mathbb{Z}_\ell \times \mathbb{Z}_m$, and $N_\alpha^{[1]} \cong \mathbb{Z}_\ell$, where $\ell \mid m$, $m \mid (p^d - 1)$ and $d \mid f$, or*
  - (b) *Γ is N-locally primitive, and Lemma 5.4(iv) is satisfied.*

The theorem has the following simpler version.

Corollary 1.2. *Let Γ be a connected $(G, 2)$-arc-transitive graph, and let N be a vertex-transitive normal subgroup of G. Then at least one of the following holds:*