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On the uniqueness of the generalized octagon of order (2, 4) ☆

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ABSTRACT

The smallest known thick generalized octagon has order $(2,4)$ and can be constructed from the parabolic subgroups of the Ree group ${}^2F_4(2)$. It is not known whether this generalized octagon is unique up to isomorphism. We show that it is unique up to isomorphism among those having a point a whose stabilizer in the automorphism group both fixes setwise every line on a and contains a subgroup that is regular on the set of 1024 points at maximal distance to a. Our proof uses extensively the classification of the groups of order dividing 2^9.

Dedicated to the memory of Ákos Seress

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1. Introduction

Recall that a point-line geometry $\mathcal{G} = (P, \mathcal{L})$ consists of a set P of *points*, and a collection $\mathcal{L}$ of subsets of P, each of size at least two, called *lines*. A point $p \in P$ is *incident* with a line $L \in \mathcal{L}$ if p is an element of L. Two points incident to the same line are *collinear*. We associate two graphs with every point-line geometry. The *incidence graph* of $\mathcal{G}$ has as vertices all points and all lines of $\mathcal{G}$, with edges connecting incident point-line pairs. The *collinearity graph* of $\mathcal{G}$ has as vertices all points of $\mathcal{G}$, with edges connecting collinear pairs of points. It is easy to see that the incidence graph is connected if and only if the collinearity graph is connected; if so, then $\mathcal{G}$ is *connected*.

The *diameter* of a connected graph Γ is the maximal distance between vertices and its *girth* is the shortest length of a cycle.

Definition 1.1. For $n \geq 3$, a *generalized n-gon* is a point-line geometry $\mathcal{G}$ satisfying the following properties:

(1) the diameter of the incidence graph Γ of $\mathcal{G}$ is n;
(2) the girth of Γ is $2n$;
(3) $\mathcal{G}$ is regular: every point is incident with the same number $t + 1 > 1$ of lines and every line is incident with the same number $s + 1 > 1$ of points.

This concept was introduced and developed by Tits [13,14].

We are interested in finite generalized n-gons when both s and t are finite. The pair (s, t) is the *order* of $\mathcal{G}$. The generalized n-gon $\mathcal{G}$ is *thin* if $s = 1 = t$. For every n there exists exactly one thin generalized n-gon, which can be described as the geometry of all vertices and edges of the usual n-gon.

By a famous theorem of Feit and Higman [10], a finite generalized n-gon with $n \geq 3$ is either thin, or satisfies $n \in \{3, 4, 6, 8, 12\}$. Furthermore, if $\mathcal{G}$ is *thick* (that is, both $s, t \geq 2$), then $n = 12$ is impossible. Thus the *gonality* n of a thick finite generalized n-gon is at most eight. This largest gonality, $n = 8$, is the only case where the smallest thick generalized n-gons are not known up to isomorphism. The smallest thick finite generalized quadrangles and hexagons are unique (see [7] and [12]).

It follows from [10] that the smallest order for which a thick finite generalized octagon can exist is $(2, 4)$. A generalized octagon of order $(2, 4)$ was constructed by Tits [15] as part of an infinite series of generalized octagons related to the groups ${}^2\mathrm{F}_4(q)$. The octagon of order $(2, 4)$ is obtained from ${}^2\mathrm{F}_4(2)$ by taking for points the maximal parabolic subgroups G_1 for which $G_1/O_2(G_1)$ is a Frobenius group of order 20, and by taking for lines the maximal parabolics G_2 of the other type, with incidence between G_1 and G_2 defined by $O_2(G_1) \subseteq G_2$; more details are given in Example 1.3.

The uniqueness, or otherwise, of the generalized octagon of order $(2, 4)$ remains an open and very difficult problem. In [8], De Bruyn proved that the example of Tits is the only one in which the unique generalized octagon of order $(2, 1)$ embeds. In [3], it