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On the parameterized differential inverse Galois problem over $k((t))(x)$

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ABSTRACT

In this article, we consider the inverse Galois problem for parameterized differential equations over $k((t))(x)$ with k any field of characteristic zero and use the method of patching over fields due to Harbater and Hartmann. We show that every connected semisimple $k((t))$-split linear algebraic group is a parameterized differential Galois group over $k((t))(x)$.

Introduction

In classical Galois theory, we are given a polynomial f over a field F and consider the field E obtained by adjoining all roots of f inside an algebraic closure. The Galois group is the finite group of all automorphisms of E that act trivially on F. The inverse problem asks which finite groups are Galois groups over a given field F. For example

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over $\mathbb{Q}$ this is still an open problem. In differential Galois theory, we start with a linear differential equation over a differential field F and look at the automorphisms of the field obtained by adjoining a complete set of solutions that act trivially on F and commute with the derivation. This group measures the algebraic relations among the solutions. Parameterized differential Galois theory is a refinement of differential Galois theory where the Galois group measures algebraic relations as well as the ∂_t-algebraic relations among the solutions, if the base field F is equipped with an additional derivation ∂_t depending on a parameter t.

Let k be a field of characteristic zero and define $F = k((t))(x)$ with the two natural derivations ∂_x and ∂_t. Consider a linear differential equation $\partial_x(y) = Ay$ for some matrix $A \in F^{n\times n}$ (the set of all $n \times n$-matrices over F). The Galois theory of parameterized differential equations as introduced in [3] assigns a parameterized differential Galois group to this equation which can be identified with a linear differential group $\mathcal{G} \leq \mathrm{GL}_n$ defined over $k((t))$. That means that $\mathcal{G} \leq \mathrm{GL}_n$ is given by ∂_t-differential polynomials in the coordinates of GL_n over $k((t))$. The inverse Galois problem in this situation asks which linear differential groups defined over $k((t))$ can be obtained as parameterized differential Galois groups of some differential equations.

So far, the inverse Galois problem for parameterized differential equations has only been considered over $U(x)$ where U is equipped with a derivation ∂_t (or more generally with several commuting derivations $\partial_{t_1}, \dots, \partial_{t_r}$) and is differentially closed or even stronger, a universal differential field. This means that for any differential field $L \subseteq U$ which is finitely differentially generated over $\mathbb{Q}$, any differentially finitely generated extension of L can be embedded into U. Over $U(x)$ for such a field U, the following necessary [4] and sufficient [10] condition was recently found: A linear differential group $\mathcal{G}$ defined over U is a parameterized differential Galois group if and only if $\mathcal{G}$ is differentially finitely generated, that is, if there are finitely many elements $g_1, \dots, g_m \in \mathcal{G}(U)$ such that $\mathcal{G}(U)$ is the smallest closed differential subgroup of $\mathrm{GL}_n(U)$ containing them. In the special case of a linear algebraic group $\mathcal{G}$ over U, Singer then showed that $\mathcal{G}$ is differentially finitely generated if and only if the identity component $\mathcal{G}^\circ$ has no quotient isomorphic to the additive group $\mathbb{G}_a$ or the multiplicative group $\mathbb{G}_m$ [11]. This implies in particular that every semisimple linear algebraic group defined over U is a parameterized differential Galois group over $U(x)$. For unipotent and reductive linear differential algebraic groups, there are also recent results that give characterizations which groups are finitely generated as differential algebraic groups [9,8].

Over fields $U(x)$ with U not differentially closed, not much is known on the inverse problem. We restrict ourselves to the base field $F = k((t))(x)$ as above. This is the function field of a curve over a complete discretely valued field. Over such a field we can apply the method of patching over fields due to Harbater and Hartmann (see Theorem 2.1). This method has been applied by Harbater and Hartmann to (non-parameterized) differential Galois theory. In [6] they show that any linear algebraic group that is generated by finitely many subgroups $\mathcal{G}_1, \dots, \mathcal{G}_r$ such that each $\mathcal{G}_i$ is either finite or $k((t))$-isomorphic to $\mathbb{G}_a$ or $\mathbb{G}_m$ is the differential Galois group of some differential equation over $k((t))(x)$.