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The geometry of special symplectic representations

Marcus J. Slupinski [a,*], Robert J. Stanton [b]

[a] *IRMA, Université de Strasbourg, 7 rue René Descartes, F-67084 Strasbourg cedex, France*
[b] *Department of Mathematics, Ohio State University, 231 West 18th Avenue, Columbus, OH 43210-1174, United States*

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ABSTRACT

We show there is a class of symplectic Lie algebra representations over any field of characteristic not 2 or 3 that have many of the exceptional algebraic and geometric properties of both symmetric three forms in two dimensions and alternating three forms in six dimensions. The main algebraic result is that suitably generic elements of these representation spaces can be uniquely written as the sum of two elements of a naturally defined Lagrangian subvariety. We give universal explicit formulae for the summands and show how they lead to the existence of geometric structure on appropriate subsets of the representation space. Over the real numbers this structure reduces to either a conic, special pseudo-Kähler metric or a conic, special para-Kähler metric.

1. Introduction

It has been known since the mid 19th century that symmetric three forms in two dimensions (binary cubics) possess remarkable algebraic properties. More recently [13] it was shown that real alternating three forms in six dimensions also have special algebraic

* Corresponding author.
E-mail addresses: marcus.slupinski@math.unistra.fr (M.J. Slupinski), stanton@math.ohio-state.edu (R.J. Stanton).

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properties. A common feature of these two spaces is that they are quite naturally symplectic vector spaces and there is a natural choice of Lie algebra acting symplectically on them. In the case of real three forms, Hitchin [13] exploited this observation extensively, and, although historically the symplectic aspect with regard to binary cubics has been largely ignored, we [20] showed that many of their important properties can be expressed in purely symplectic terms. The main purpose of this paper is to show that there is a class of symplectic Lie algebra representations over any field k of characteristic not 2 or 3 that have many of the remarkable algebraic and geometric properties of both binary cubics and alternating three forms in six dimensions. These representations, which we call special symplectic representations (SSR), are necessarily rare but, besides the aforementioned two examples, include:

- $\mathfrak{sp}(6,k)$ acting on primitive alternating three forms in six dimensions;
- a k-form of $\mathfrak{so}(12,\bar{k})$ acting in a half-spinor representation defined over k;
- a k-form of $\mathfrak{e}_7$ acting in its familiar 56 dimensional representation;
- specific representations of k-forms of classical Lie algebras, in particular cf. Appendix A.

The real vector space of alternating three forms in six dimensions is doubly interesting because its special algebraic properties have geometric implications. For example, certain of its open subsets are naturally endowed with “special” differential geometric structure (e.g. a special pseudo-Kähler metric [13]). In the last section of the paper we show that the special algebraic properties of all SSRs lead to special geometry in all cases so that, suitably interpreted algebro-geometrically, this differential geometric structure exists for special symplectic representations over a field k of characteristic not 2 or 3.

A special symplectic representation is a symplectic Lie algebra representation with extra structure. If $\mathfrak{m}$ denotes the Lie algebra and V the representation space, this extra structure is an equivariant quadratic map (a co-moment map) $\mu : V \to \mathfrak{m}$ satisfying a constraint (cf. Definition 2.1). From μ and the symplectic form ω one can form two other symplectic covariants $\Psi : V \to V$ and $Q : V \to k$, and these are the main technical tools of the paper. The three symplectic covariants generalise to any SSR the classical covariants of a binary cubic defined by Eisenstein [8].

Our first results describe orbit properties of SSRs or, more precisely, properties of the vector space $\mathfrak{m} \cdot v$, the tangent space to a group orbit through $v \in V$ if the action of $\mathfrak{m}$ is integrable. An unusual and important property is that $\mathfrak{m} \cdot v$ is coisotropic if $v \neq 0$, and we think this property may characterise SSRs. For binary cubics it is more or less evident but seems not to have been known for other SSRs. Of particular interest are generic orbits and minimal orbits. We show that $\mathfrak{m} \cdot v$ is of codimension one if $Q(v) \neq 0$ and that $\mathfrak{m} \cdot v$ is Lagrangian iff $\mu(v) = 0$ and $v \neq 0$. In particular, an SSR is a prehomogeneous vector space for the Lie algebra $k \times \mathfrak{m}$ and the open orbits provide examples of noncommutative, completely integrable systems if $k = \mathbb{R}$ or $k = \mathbb{C}$. Although prehomogeneous vector spaces for algebraic groups have been very widely studied in the