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Relative degrees of irreducible morphisms

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ABSTRACT

We answer affirmatively a recently asked question in [5] which relates the degree of an irreducible morphism to the composition of a path of irreducible morphisms between indecomposable modules. We introduce the notion of the degree of an irreducible morphism related to a subcategory $\mathfrak{D}$ of ind$A$ and we study this notion in case $\mathfrak{D}$ is an element of the postprojective partition $\{\mathbf{P_0}, \cdots, \mathbf{P_n}, \cdots, \mathbf{P_\infty}\}$ of ind$A$. We also present a few connections between degrees of irreducible morphisms and the theory of postprojective partition.

Let A be a basic finite dimensional algebra over an algebraically closed field k. The notion of left (dually, right) degree of an irreducible morphism was introduced by Liu in [8] and applied in the same paper to obtain information about the possible shapes of connected components of the Auslander–Reiten quiver of an Artin algebra. We say that the left degree of an irreducible morphism $f : X \to Y$ is n, and we denote $d_l(f) = n$, if n is the smallest positive integer for which there exist $Z \in \text{ind}A$ and a morphism

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$h : Z \to X$ such that $h \in \mathrm{rad}^n(Z, X)\backslash\mathrm{rad}^{n+1}(Z, X)$ and $fh \in \mathrm{rad}^{n+2}(Z, Y)$. In case this condition is not verified for any $n \geq 1$ we say the left degree of f is infinite.

An important result was obtained recently in [5] for an algebra A, namely if f and h are as above and $d_l(f) = n$ we can always replace h by a morphism h' in $\mathrm{rad}^n(Z, X)\backslash\mathrm{rad}^{n+1}(Z, X)$ such that $fh' = 0$. It was asked in [5] (Remark 3.1) if given an irreducible morphism $f : X \to Y$ with $X \in \mathrm{ind}A$, $Z \in \mathrm{ind}A$ and a morphism $h \in \mathrm{rad}^n(Z, X)\backslash\mathrm{rad}^{n+1}(Z, X)$ such that $fh = 0$, it is possible to choose h to be a composition of irreducible morphisms between indecomposable modules. This question was motivated by the fact that the degree of an irreducible morphism always seemed to be related to the behavior of the composition of irreducible morphisms between indecomposable modules as one can see from the special cases already obtained as mentioned in the same remark (see also Theorem 3.4 in [4]).

In this paper we give an affirmative answer to this problem (Theorem 3.3) in a more general context which is provided by the notion introduced here of left degree of an irreducible morphism relative to a subcategory $\mathfrak{D}$ of $\mathrm{ind}A$. In fact, we also require the stronger condition that the path of irreducible morphisms which replaces h above is of length n (it could be smaller). For this purpose, it was necessary to make use of the generic covering introduced in [5].

The notion of left degree of an irreducible morphism relative to a subcategory $\mathfrak{D}$ of $\mathrm{ind}A$ also becomes interesting when we study the connections between the degrees of irreducible morphisms and the postprojective partition $\{\mathbf{P_0}, \cdots, \mathbf{P_n}, \cdots, \mathbf{P}_\infty\}$ of $\mathrm{ind}A$ as introduced by Auslander and Smalø in [1]. For instance, if the subcategory $\mathfrak{D}$ is the category of indecomposable projective modules $\mathbf{P_0}$ then we present a characterization (Theorem 4.5) of whether the left degree of an irreducible morphism relative to $\mathfrak{D}$ is finite or infinite which does not rely on the concept of radical.

This paper is organized as follows. After recalling some basic notions in Section 1, we introduce in Section 2 the concept of degrees of irreducible morphisms relative to subcategories. Sections 3 and 4 are then devoted to the proof of the main results.

1. Preliminaries

1.1. Quivers

A *quiver* is given by two sets Γ_0 and Γ_1 together with two maps $s, e : \Gamma_1 \to \Gamma_0$. The elements of Γ_0 are called *vertices* and the elements of Γ_1 are called *arrows*. A quiver Γ is said to be *locally finite* if each vertex of Γ_0 is the starting and the ending point of at most finitely many arrows in Γ_1.

A pair (Γ, τ) is said to be a *translation quiver* provided Γ is a quiver without loops (that is, arrows starting and ending at the same vertex) and locally finite and $\tau : \Gamma_0' \to \Gamma_0''$ is a bijection characterized as follows: Γ_0' and Γ_0'' are both subsets of Γ_0 and, for every $x \in \Gamma_0$ such that τx exists, there exists a bijection from the set x^- of arrows arriving at x to the set $(\tau x)^+$ of arrows starting at τx. The vertices of Γ which are not in Γ_0' are