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Finite generation of Lie algebras associated with associative algebras [☆]

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Let F be a field of characteristic not 2. An associative F-algebra R gives rise to the commutator Lie algebra $R^{(-)} = (R, [a,b] = ab - ba)$. If the algebra R is equipped with an involution $* : R \to R$ then the space of the skew-symmetric elements $K = \{a \in R \mid a^* = -a\}$ is a Lie subalgebra of $R^{(-)}$. In this paper we find sufficient conditions for the Lie algebras $[R,R]$ and $[K,K]$ to be finitely generated.

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1. Introduction

Let F be a field of characteristic not 2. An associative F-algebra R gives rise to the commutator Lie algebra $R^{(-)} = (R, [a,b] = ab - ba)$ and the Jordan algebra $R^{(+)} = (R, a \circ b = \frac{1}{2}(ab + ba))$. If the algebra R is equipped with an involution $* : R \to R$ then the space of skew-symmetric elements $K = \{a \in R \mid a^* = -a\}$ is a Lie subalgebra of

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$R^{(-)}$, the space of symmetric elements $H = \{a \in R \mid a^* = a\}$ is a Jordan subalgebra of $R^{(+)}$. Following the result of J.M. Osborn (see [4]) on finite generation of the Jordan algebras $R^{(+)}$, H. I. Herstein [4] raised the question about the finite generation of Lie algebras associated with R. In this paper we find sufficient conditions for the Lie algebras $[R^{(-)}, R^{(-)}]$, $[K, K]$ to be finitely generated.

Theorem 1. *Let R be a finitely generated associative F-algebra with an idempotent e such that $ReR = R(1-e)R = R$. Then the Lie algebra $[R, R]$ is finitely generated.*

The following example shows that the idempotent condition can not be dropped.

Example 1. The algebra $R = \begin{pmatrix} F[x] & F[x] \\ 0 & F[x] \end{pmatrix}$ of triangular 2×2 matrices over the polynomial algebra $F[x]$ is finitely generated. However the Lie algebra $[R, R] = \begin{pmatrix} 0 & F[x] \\ 0 & 0 \end{pmatrix}$ is not.

Theorem 2. *Let R be a finitely generated associative F-algebra with an involution $* : R \to R$. Suppose that R contains an idempotent e such that $ee^* = e^*e = 0$ and $ReR = R(1-e-e^*)R = R$. Then the Lie algebra $[K, K]$ is finitely generated.*

The following example shows that the condition on the idempotent cannot be relaxed.

Example 2. Consider the associative commutative algebra $A = F[x, y]/id(x^2)$ with the automorphism φ of order 2: $\varphi(x) = -x$, $\varphi(y) = y$. The algebra $R = M_2(A)$ of 2×2 matrices over A has an involution $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \to \begin{pmatrix} d^\varphi & b^\varphi \\ c^\varphi & a^\varphi \end{pmatrix}$. We have $[K, K] \leq xM_2(F[y])$, $dim_F[K, K] = \infty$, which implies that algebra $[K, K]$ is not finitely generated.

W.E. Baxter [2] showed that if R is a simple F-algebra, which is not $\leq$16 dimensional over its center Z then the Lie algebra $[K, K]/[K, K] \cap Z$ is simple.

Theorem 3. *Let R be a simple finitely generated F-algebra with an involution $* : R \to R$. Suppose that R contains an idempotent e such that $ee^* = e^*e = 0$. Then the Lie algebra $[K, K]/[K, K] \cap Z$ is finitely generated.*

2. Finite generation of Lie algebras $[R, R]$

Consider the Peirce decomposition $R = eRe + eR(1-e) + (1-e)Re + (1-e)R(1-e)$. The components $eR(1-e)$, $(1-e)Re$ lie in $[R, R]$ since $eR(1-e) = [e, eR(1-e)]$, $(1-e)Re = [e, (1-e)Re]$.

Lemma 1. *The Lie algebra $[R, R]$ is generated by $eR(1-e) + (1-e)Re$.*

Proof. We only need to show that

$$[eRe, eRe] + \big[(1-e)R(1-e), (1-e)R(1-e)\big] \subseteq \text{Lie}\,\big\langle eR(1-e), (1-e)Re\big\rangle.$$

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