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# Restriction theorems for principal bundles in arbitrary characteristic

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ABSTRACT

The aim of this paper is to give a proof of the restriction theorems for principal bundles with a reductive algebraic group as structure group in arbitrary characteristic. Let $G$ be a reductive algebraic group over any field $k = \bar{k}$, let $X$ be a smooth projective variety over $k$, let $H$ be a very ample line bundle on $X$ and let $E$ be a semistable (resp. stable) principal $G$-bundle on $X$ w.r.t. $H$. The main result of this paper is that the restriction of $E$ to a general smooth curve which is a complete intersection of ample hypersurfaces of sufficiently high degrees is again semistable (resp. stable).


## 1. Introduction

Around 1981–1982, V. Mehta and A. Ramanathan proved the following important theorem (see [3]): Let $X$ be a smooth projective variety over $k = \bar{k}$ with a chosen polarisation and let $E$ be a torsion free coherent sheaf on it. Then the restriction of $E$ to a general, non-singular curve which is a complete-intersection of ample hypersurfaces of sufficiently high degrees is again semistable. V. Mehta and A. Ramanathan later also gave

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a proof of the stable restriction theorem for torsion-free coherent sheaves (see [4]). Later, an effective restriction theorem for torsion-free coherent sheaves on normal projective varieties over algebraically closed fields of characteristic zero was proved by H. Flenner (see [7]). An effective semistable restriction theorem for principal bundles with a reductive algebraic group as structure group over smooth, projective varieties defined over algebraically closed fields of characteristic zero follows immediately since the semistability of a principal $G$-bundle in characteristic zero is equivalent to the semistability of its adjoint bundle. The technique of Flenner was used by Biswas and Gomez (see [8]) to obtain another effective restriction theorem for principal bundles over algebraically closed fields of characteristic zero. They also proved a restriction theorem for stable bundles.

A version of the restriction theorem for strongly semistable bundles exists in [10]. In another direction A. Langer proved an effective restriction theorem for torsion-free sheaves in positive characteristic (see [9]). However the semistable and stable restriction theorem for principal bundles remain open in positive characteristic. The aim of this paper is to prove these two restriction theorems. The basic idea of the proofs is similar to that of the semistable restriction theorem in [3]. The proofs given here are characteristic free. The paper is arranged as follows:

In Section 2, we introduce some preliminary notions and set up some notations which will be used throughout the paper.

In Section 3 we recall a degeneration argument which is central to the proofs in [3] and [4] and also draw some consequences out of it.

In Section 4 we prove the semistable restriction theorem for principal bundles using the degeneration argument introduced in Section 3.

In Section 5 we prove the stable restriction theorem for principal bundles analogous to the stable restriction theorem for torsion-free coherent sheaves proved in [4]. The proof given here is different and substantially simpler than the proof of the stable restriction theorem in [4].

## 2. Preliminaries

In this section we set up some notation and recall some basic facts which will be used in the paper. Many of these have been taken from [1] with only minor changes. $X$ will always stand for a smooth projective variety defined over a field $k = \bar{k}$ of arbitrary characteristic. $H$ will denote the chosen polarisation on $X$. Let $G \supset B \supset T$ be a reductive group, together with a chosen Borel subgroup and a maximal torus. As usual, $X^*(T)$ and $X_*(T)$ will respectively denote the groups of all characters and all 1-parameter subgroups of $T$. We choose once for all, a Weyl group invariant positive definite bilinear form on $\mathbb{Q} \otimes X^*(T)$. This, in particular, will allow us to identify $\mathbb{Q} \otimes X^*(T)$ with $\mathbb{Q} \otimes X_*(T)$. Let $\Delta \subset X^*(T)$ be the corresponding simple roots. Let $\omega_\alpha \in \mathbb{Q} \otimes X^*(T)$ denote the fundamental dominant weight corresponding to $\alpha \in \Delta$, so that $\langle \omega_\alpha, \beta^\vee \rangle = \delta_{\alpha,\beta}$ where $\beta^\vee \in \mathbb{Q} \otimes X_*(T)$ is the simple coroot corresponding to $\beta \in \Delta$. Note that each $\omega_\alpha$ is a non-negative rational linear combination of the simple roots $\alpha$. Recall that *the closed*