Journal of Algebra 426 (2015) 92–136

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Universal enveloping algebras of Poisson Hopf algebras

Jiafeng Lü [a], Xingting Wang [b], Guangbin Zhuang [c,*]

[a] *Department of Mathematics, Zhejiang Normal University, Jinhua 321004, China*
[b] *Department of Mathematics, University of California, San Diego, CA 92093, USA*
[c] *Department of Mathematics, University of Southern California, Los Angeles, CA 90089-2532, USA*

ARTICLE INFO

Article history:
Received 8 April 2014
Available online 7 January 2015
Communicated by Nicolás Andruskiewitsch

MSC:
17B63
16T05
16S10

Keywords:
Hopf algebras
Poisson Hopf algebras
Universal enveloping algebras
Lie–Rinehart algebras

ABSTRACT

For a Poisson algebra A, by exploring its relation with Lie–Rinehart algebras, we prove a Poincaré–Birkhoff–Witt theorem for its universal enveloping algebra A^e. Some general properties of the universal enveloping algebras of Poisson Hopf algebras are studied. Given a Poisson Hopf algebra B, we give the necessary and sufficient conditions for a Poisson polynomial algebra $B[x; \alpha, \delta]_p$ to be a Poisson Hopf algebra. We also prove a structure theorem for B^e when B is a pointed Poisson Hopf algebra. Namely, B^e is isomorphic to $B\#_\sigma \mathcal{H}(B)$, the crossed product of B and $\mathcal{H}(B)$, where $\mathcal{H}(B)$ is the quotient Hopf algebra B^e/B^eB^+.

1. Introduction

The notion of Poisson Hopf algebras arises naturally in the study of Poisson geometry and quantum groups. For example, the coordinate ring of a Poisson algebraic group

* Corresponding author.
E-mail addresses: jiafenglv@gmail.com (J. Lü), xiw199@ucsd.edu (X. Wang), gzhuang@usc.edu (G. Zhuang).

http://dx.doi.org/10.1016/j.jalgebra.2014.12.010
0021-8693/© 2014 Published by Elsevier Inc.

is a Poisson Hopf algebra [10, Definition 3.1.6]. Recently, connected Hopf algebras are studied in a series of papers [4,24,23,26]. In [26], the third named author showed that, for a connected Hopf algebra H, the associated graded algebra $\operatorname{gr} H$ with respect to the coradical filtration is always commutative. Therefore $\operatorname{gr} H$ carries an induced Poisson structure, which actually makes $\operatorname{gr} H$ into a Poisson Hopf algebra. This observation suggests, in a very rough sense, that one should think of connected Hopf algebras as deformations of Poisson Hopf algebras. While the study of Poisson Hopf algebras is interesting on its own, we certainly hope that it will help us to understand Hopf algebras in general.

For a Poisson algebra A, one can define its universal enveloping algebra A^e, which basically transfers the Poisson structure of A to the algebra structure of A^e. Generally speaking, the algebra A^e could be very complicated and highly non-commutative. For example, in [21], Umirbaev proved that the universal enveloping algebras of the Poisson symplectic algebra P_n is isomorphic to $A_n \otimes A_n^{op}$, where A_n is the n-th Weyl algebra. In [11], the authors of this paper proved that any Poisson–Ore extension of A induced an iterated Ore extension of A^e. If we start with a Poisson Hopf algebra B, then B^e is actually a Hopf algebra [13], whose module category is equivalent to the category of Poisson modules over B as monoidal categories. Hence, from a categorical point of view, the Hopf algebra B^e carries all information we need to understand the Poisson Hopf algebra B, which motivates us to study the structure of B^e.

The paper is organized as follows.

In Section 2 and Section 3, we briefly review some basic concepts related to Poisson Hopf algebras and give some examples. Section 4 is devoted to the study of Poisson polynomial algebras. To be specific, for a given Poisson Hopf algebra B, we give the necessary and sufficient conditions for a Poisson polynomial algebra $B[x;\alpha,\delta]_p$ to be a Poisson Hopf algebra.

In Section 5, we review the definition and some basic properties of A^e, the enveloping algebra of a Poisson algebra A. A well-known result in [8] states that the Kälher differential Ω_A of a Poisson algebra is a Lie–Rinehart algebra. By using the universal property of A^e [13], we show that A^e is isomorphic to $V(A,\Omega_A)$, the enveloping algebra of the Lie–Rinehart algebra Ω_A. Consequently, a Poincaré–Birkhoff–Witt theorem for A^e is derived.

In the rest of the sections we focus on the structure of B^e where B is a Poisson Hopf algebra. In [13], it is shown that B^e is a Hopf algebra with B as a Hopf subalgebra. By using standard techniques from Hopf algebra theory, we show in Section 6 that B^e is always a free module over B and an injective comodule over $\mathcal{H}(B)$, where $\mathcal{H}(B) = B^e/B^eB^+$. Moreover, when B is a pointed Poisson algebra, we are able to prove that

$$B^e \cong B\#_\sigma\mathcal{H}(B)$$

as algebras, where $B\#_\sigma\mathcal{H}(B)$ is the crossed product of B and $\mathcal{H}(B)$. This result involves showing that the extension $B \subset B^e$ has the normal basis property and is Galois. In