Journal of Algebra 426 (2015) 188–210

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

The Grothendieck group of non-commutative non-noetherian analogues of $\mathbb{P}^1$ and regular algebras of global dimension two

Gautam Sisodia, S. Paul Smith *

Department of Mathematics, Box 354350, Univ. Washington, Seattle, WA 98195, United States

ARTICLE INFO

Article history:
Received 6 March 2014
Available online 9 January 2015
Communicated by Louis Rowen

MSC:
16W50
16E65
19A99

Keywords:
Regular algebras
Graded rings
Global dimension
Grothendieck group

ABSTRACT

Let V be a finite-dimensional positively-graded vector space. Let $b \in V \otimes V$ be a homogeneous element whose rank is $\dim(V)$. Let $A = TV/(b)$, the quotient of the tensor algebra TV modulo the 2-sided ideal generated by b. Let $\mathsf{gr}(A)$ be the category of finitely presented graded left A-modules and $\mathsf{fdim}(A)$ its full subcategory of finite dimensional modules. Let $\mathsf{qgr}(A)$ be the quotient category $\mathsf{gr}(A)/\mathsf{fdim}(A)$. We compute the Grothendieck group $K_0(\mathsf{qgr}(A))$. In particular, if the reciprocal of the Hilbert series of A, which is a polynomial, is irreducible, then $K_0(\mathsf{qgr}(A)) \cong \mathbb{Z}[\theta] \subset \mathbb{R}$ as ordered abelian groups where θ is the smallest positive real root of that polynomial. When $\dim_k(V) = 2$, $\mathsf{qgr}(A)$ is equivalent to the category of coherent sheaves on the projective line, $\mathbb{P}^1$, or a stacky $\mathbb{P}^1$ if V is not concentrated in degree 1. If $\dim_k(V) \geq 3$, results of Piontkovski and Minamoto suggest that $\mathsf{qgr}(A)$ behaves as if it is the category of "coherent sheaves" on a non-commutative, non-noetherian analogue of $\mathbb{P}^1$.

* Corresponding author.
E-mail addresses: gautas@math.washington.edu (G. Sisodia), smith@math.washington.edu (S.P. Smith).

http://dx.doi.org/10.1016/j.jalgebra.2014.11.024
0021-8693/

1. Introduction

1.1. Let k be a field. Let $A = TV/(b)$ where V is a finite-dimensional positively-graded k-vector space and $b \in V \otimes V$ a homogeneous element of rank $\dim(V)$. Zhang [16] showed that up to equivalence the category $\mathsf{gr}(A)$ depends only on V as a graded vector space, not on b.

This paper is motivated by non-commutative algebraic geometry. As we explain in Section 1.3, results of Piontkovski and Minamoto suggest that the category $\mathsf{qgr}(A)$ behaves as if it is the category of "coherent sheaves" on a non-commutative, non-noetherian if $\dim_k(V) \geq 3$, analogue of the projective line, $\mathbb{P}^1$. With this perspective we are computing the Grothendieck groups of "coherent sheaves" on these non-commutative analogues of $\mathbb{P}^1$.

The description of $K_0(\mathsf{qgr}(A))$ as an *ordered* abelian group says that $\alpha \in \mathbb{Z}[\theta] \cong K_0(\mathsf{qgr}(A))$ is equal to $[\mathcal{F}]$ for some $\mathcal{F} \in \mathsf{qgr}(A)$ if and only if $\alpha \geq 0$.

The result reminds us of the fact that the K_0 of an irrational rotation algebra $\mathcal{A}_\theta$ is isomorphic to $\mathbb{Z}[\theta] \subseteq \mathbb{R}_{\geq 0}$ as an ordered abelian group [11,13]. We do not know any connection between $\mathsf{qgr}(A)$ and $\mathcal{A}_\theta$.

1.2. The case $\dim_k(V) = 2$

When $V = kx_0 \oplus kx_1$ with $\deg(x_0) = \deg(x_1) = 1$ and $b = x_0x_1 - x_1x_0$, A is the polynomial ring $k[x_0, x_1]$ and the category $\mathsf{qgr}(A)$ of finitely generated graded A-modules modulo the full subcategory of finite dimensional graded A-modules is equivalent to the category $\mathsf{coh}(\mathbb{P}^1)$ of coherent sheaves on $\mathbb{P}^1$. If $\deg(x_0) = 1$ and $\deg(x_1) = m > 1$ and $b = x_0x_1 - x_1x_0$, then $\mathsf{qgr}(A)$ is equivalent to $\mathsf{coh}[\mathbb{P}^1/\mathbb{Z}_m]$, the coherent sheaves on the stacky $\mathbb{P}^1$ with a single stacky point isomorphic to $B\mathbb{Z}_m$. The K_0 of this stack, and more general toric DM stacks, is computed in [14].

1.3. In [12], Piontkovski shows that $A = TV/(b)$ behaves like a homogeneous coordinate ring of a non-commutative (non-noetherian if $n \geq 3$) analogue of the projective line. He proves that A is graded coherent and hence that the category $\mathsf{qgr}(A)$ of finitely presented graded A-modules modulo the full subcategory of finite dimensional graded A-modules is abelian category. He shows that $\mathsf{qgr}(A)$ is like $\mathsf{coh}(\mathbb{P}^1)$ in so far as it has cohomological dimension 1, Ext groups have finite dimension, and satisfies Serre duality. Explicitly, if $\mathcal{F}, \mathcal{G} \in \mathsf{qgr}(A)$, then $\operatorname{Ext}^2_{\mathsf{qgr}(A)}(\mathcal{F}, \mathcal{G}) = 0$, $\dim_k \operatorname{Ext}^*_{\mathsf{qgr}(A)}(\mathcal{F}, \mathcal{G}) < \infty$, and $\operatorname{Hom}_{\mathsf{qgr}(A)}(\mathcal{F}, \mathcal{G}) \cong \operatorname{Ext}^1_{\mathsf{qgr}(A)}(\mathcal{G}, \mathcal{F}(-n-1))^*$.

In [8], Minamoto gives additional evidence that $\mathsf{qgr}(A)$ is like $\mathsf{coh}(\mathbb{P}^1)$ by proving an analogue of the well-known equivalence, $\mathsf{D}^b(\mathsf{Qcoh}(\mathbb{P}^1_k)) \equiv \mathsf{D}^b(kQ_2)$, of bounded derived categories where kQ_2 is the path algebra of the quiver with two vertices and 2 arrows from the first vertex to the second. Minamoto shows that $\mathsf{D}^b(\mathsf{Qcoh}(A)) \equiv \mathsf{D}^b(kQ_{n+1})$ where Q_{n+1} is the path algebra of the quiver with two vertices and $n+1$ arrows from the first vertex to the second.