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Long length functions

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ABSTRACT

Let D be an integral domain satisfying ACCP. We refine the classical notion of (factorization) length by recursively defining the *length* of a nonzero element to be the least ordinal strictly greater than the lengths of its proper divisors. This gives a surjective function $L : D^* \to L(D)$, where $L(D)$, called the *length* of D, is the least ordinal strictly greater than the length of any nonzero element. We show that an ordinal is the length of a domain satisfying ACCP if and only if it is of the form ω^β. We give some conditions for when monoid domains, generalized power series domains, inert extensions, or localizations at splitting sets satisfy ACCP, and calculate the lengths of these domains in these cases. Finally, for each positive integer $n \geq 2$ and each ordinal $\mu \geq n$, we construct a domain D satisfying ACCP and an $x \in D^*$ with $L(x) = \mu$ and $l(x) = n$, where $l(x)$ denotes the number of factors in a minimum length atomic factorization of x.

Throughout this paper, all rings will be commutative with $1 \neq 0$, and all monoids will be commutative. Let D be an integral domain with quotient field K. As usual,

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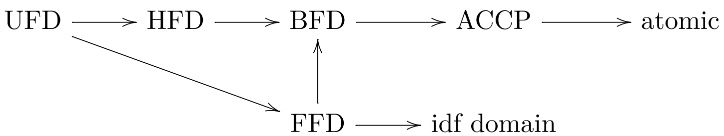
$D^* = D\backslash\{0\}$, $D^\times$ denotes the group of units of D, and $D^\# = D^*\backslash D^\times$. By a *factorization* of *length* n of $x \in D^\#$, we mean an expression $x = x_1 \cdots x_n$ with $n \geq 1$ and each $x_i \in D^\#$. An *atom* is a nonzero nonunit with no factorization of length 2, and an *atomic factorization* is a factorization whose factors are atoms. A domain is called (i) *atomic* if every nonzero nonunit has an atomic factorization, (ii) a *bounded factorization domain* (*BFD*) if each nonzero nonunit has a finite upper bound on the lengths of its factorizations, (iii) a *finite factorization domain* (*FFD*) if each nonzero nonunit has only finitely many factorizations up to order and associates, (iv) a *half factorial domain* (HFD) if it is atomic and any two atomic factorizations of a given nonzero nonunit have the same length, and (v) an *idf domain* if each nonzero nonunit has only finitely many irreducible divisors up to associates. A domain is an FFD if and only if it is an atomic idf domain [2, Theorem 5.1]. It is easy to see that the following diagram of implications holds, and [2] gives examples showing that there are no further nontrivial implications.



(Here ACCP is the ascending chain condition on principal ideals.) We refer the reader to [2] for further examples and results concerning these various kinds of factorization domains. One fairly extensive general reference for factorization theory is [8].

Classically, the (factorization) length function on a domain D is the function $L : D^\# \to \mathbb{Z}^+ \cup \{\infty\}$ that takes an element to the supremum of the lengths of its factorizations; one may extend this to D^* by defining $L(x) = 0$ for $x \in D^\times$. For D satisfying ACCP, we refine this notion as follows. We recursively define the (*factorization*) *length* of $x \in D^*$, denoted $L(x)$, to be the least ordinal strictly greater than the lengths of all its proper divisors. (The ACCP property is equivalent to the "proper divisor" relation being well-founded, so this recursive definition is well-defined.) We define the (*factorization*) *length* of D to be the least ordinal $L(D)$ strictly greater than the lengths of all its nonzero elements. Thus $L : D^* \to L(D)$, and we will momentarily see that L is surjective. An equivalent way to define the length function is the following. Let $P_+(D)$ be the set of nonzero principal (integral) ideals of D, and for each ordinal β recursively define L_β to be the set of maximal elements of $P_+(D) \setminus \bigcup_{\alpha<\beta} L_\alpha$. It can be verified that the L_β's form a partition of $P_+(D)$, and that $L(x) = \beta \Leftrightarrow (x) \in L_\beta$. If necessary, we will use subscripts to distinguish between length functions for different domains. Length functions can be defined analogously for cancellative monoids. Throughout this paper, we will work with the understanding that if a result for domains is proven with no use of the addition operation, then it can be generalized to cancellative monoids with essentially the same proof.