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The unitary cover of a finite group and the exponent of the Schur multiplier

Nicola Sambonet

Department of Mathematics, Technion, Haifa, 32000 Israel

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A B S T R A C T

For a finite group we introduce a particular central extension, the unitary cover, having minimal exponent among those satisfying the projective lifting property. We obtain new bounds for the exponent of the Schur multiplier relating to subnormal series, and we discover new families for which the bound is the exponent of the group. Finally, we show that unitary covers are controlled by the Zelmanov solution of the restricted Burnside problem for 2-generator groups.

1. Introduction

The *Schur multiplier* of a finite group G is the second cohomology group with complex coefficients, denoted by $\mathrm{M}(G) = H^2(G, \mathbb{C}^\times)$. It was introduced in the beginning of the twentieth century by I. Schur, aimed at the study of projective representations. To determine $\mathrm{M}(G)$ explicitly is often a difficult task. Therefore, it is of interest to provide bounds for numerical qualities of $\mathrm{M}(G)$ as the order, the rank, and – our subject – the exponent.

E-mail address: nsambonet@gmail.com.

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In 1904 Schur already showed that $[\exp \mathrm{M}(G)]^2$ divides the order of the group, and this bound is tight as $\mathrm{M}(C_n \times C_n) = C_n$. Note that $C_n \times C_n$ is an example of a group satisfying

$$\exp \mathrm{M}(G) \mid \exp G, \tag{1}$$

property which has been proven for many classes of groups.

1.1. *Groups such that* $\exp \mathrm{M}(G)$ *divides* $\exp G$

Firstly, (1) holds for every abelian group G. Indeed, consider the cyclic decomposition ordered by recursive division:

$$G = \bigoplus_{i=1}^{n} C_{d_i}, \quad d_i \mid d_{i+1}. \tag{2}$$

By Schur it is known (cf. [12, p. 317]) that

$$\mathrm{M}(G) = \bigoplus_{i=1}^{n} {C_{d_i}}^{n-i}. \tag{3}$$

Consequently, $\exp \mathrm{M}(G) = d_{n-1}$ which in turn divides $\exp G = d_n$. A second important example of groups enjoying (1) are the finite simple groups, whose multipliers are known and listed in the Atlas [4].

A standard argument (cf. [3, Th. 10.3]) proposes to focus on p-groups. Indeed, the p-component of $\mathrm{M}(G)$ is embedded in the multiplier of a p-Sylow via the restriction map. Therefore, if $\Pi(G)$ denotes the set of prime divisors of $|G|$, and S_p denotes a p-Sylow of G for $p \in \Pi(G)$, then

$$\exp \mathrm{M}(G) \mid \prod_{p \in \Pi(G)} \exp \mathrm{M}(S_p). \tag{4}$$

Clearly, since

$$\exp G = \prod_{p \in \Pi(G)} \exp S_p,$$

if (1) holds for every p-Sylow of G, then it does also for G.

A fundamental feature of p-groups is the nilpotency class. Recently, P. Moravec completed a result of M.R. Jones [11, Rem. 2.8] proving (1) for groups of class at most 3, and extended this result to groups of class 4 in the odd-order case [15, Th. 12, Th. 13]. Moravec discovered many other families enjoying (1): metabelian groups of prime exponent [14, Pr. 2.12], 3-Engel groups, 4-Engel groups in case the order is coprime to 10