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On generating series of finitely presented operads [☆]

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ABSTRACT

Given an operad P with a finite Gröbner basis of relations, we study the generating functions for the dimensions of its graded components $P(n)$. Under moderate assumptions on the relations we prove that the exponential generating function for the sequence $\{\dim P(n)\}$ is differential algebraic, and in fact algebraic if P is a symmetrization of a non-symmetric operad. If, in addition, the growth of the dimensions of $P(n)$ is bounded by an exponent of n (or a polynomial of n, in the non-symmetric case) then, moreover, the ordinary generating function for the above sequence $\{\dim P(n)\}$ is rational. We give a number of examples of calculations and discuss conjectures about the above generating functions for more general classes of operads.

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0. Introduction

We study the generating series for the dimensions of the components of algebraic operads. For symmetric operads, we conjecture that under mild restrictions these exponential generating series are differential algebraic. We prove that this is indeed the case if the operad has a finite Gröbner basis which satisfies an additional condition. Moreover, we show that if the dimensions of the components of the operad are bounded by an exponential function, then the corresponding generating function is rational. For non-symmetric operads, we show that the ordinary generating series is algebraic if the operad has a finite Gröbner basis. Moreover, the series is a rational function if, in addition, the dimensions of the operad components are bounded by a polynomial function. We also describe several algorithms for calculating the above series in various situations, and provide a number of examples of calculations. In particular, there are several natural examples of operads for which the generating series were not previously known.

0.1. Main results

Let $\mathcal{P}$ be a finitely generated operad over a field $\Bbbk$ of characteristic zero. Recall that an exponential generating series of $\mathcal{P}$ is defined as

$$E_{\mathcal{P}}(z) := \sum_{n \geqslant 1} \frac{\dim \mathcal{P}(n)}{n!} z^n. \tag{0.1.1}$$

We also consider the ordinary generating series

$$G_{\mathcal{P}}(z) := \sum_{n \geqslant 1} \dim \mathcal{P}(n) z^n. \tag{0.1.2}$$

In particular, if $\mathcal{P}$ is a symmetrization of a non-symmetric operad $\mathfrak{P}$, then $\dim \mathcal{P}(n) = n! \dim \mathfrak{P}(n)$ so that $E_{\mathcal{P}}(z) = G_{\mathfrak{P}}(z)$.

These generating series are important invariants of an operad. For example, they appear in the Ginzburg–Kapranov criterion for Koszul operads. Zinbiel [29] gives examples of such series. Moreover, the generating series for binary symmetric operads with one or two generators have been extensively studied for decades under the name of *codimension series of varieties of algebras*, see [16,4]. These are essentially the series for quotients of the operad of associative algebras.

Here we present a new approach to such series using the theory of Gröbner bases for operads developed in [10]. Note that, in general, the theory of such Gröbner bases helps answering a question: is a given operadic element equal to zero in an operad defined by a given collection of operadic relations? For example, the famous Jacobian Conjecture can be reformulated as a question of this kind [2, Section 2]. A number of operads admit a finite Gröbner basis of the ideal of relations; we refer to them as operads with a finite Gröbner basis for short. In such operads, there is an effective direct algorithm to answer