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On rational and concise words ☆

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ABSTRACT

A group-word w is called concise if whenever the set of w-values in a group G is finite it always follows that the verbal subgroup $w(G)$ is finite. More generally, a word w is said to be concise in a class of groups X if whenever the set of w-values is finite for a group $G \in X$, it always follows that $w(G)$ is finite. P. Hall asked whether every word is concise. Due to Ivanov the answer to this problem is known to be negative. It is still an open problem whether every word is concise in the class of residually finite groups. A word w is rational if the number of solutions to the equation $w(x_1, \ldots, x_k) = g$ is the same as the number of solutions to $w(x_1, \ldots, x_k) = g^e$ for every finite group G and for every e relatively prime to $|G|$. We observe that any rational word is concise in the class of residually finite groups. Further we give a sufficient condition for rationality of a word. As a corollary we deduce that the word $w = [\ldots[x_1^{n_1}, x_2]^{n_2}, \ldots, x_k]^{n_k}$ is concise in the class of residually finite groups.

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Let $w = w(x_1, \ldots, x_k)$ be a group-word, and let G be a group. The verbal subgroup $w(G)$ of G determined by w is the subgroup generated by the set G_w consisting of all values $w(g_1, \ldots, g_k)$, where $g_1, \ldots, g_k$ are elements of G. A word w is said to be concise if whenever G_w is finite for a group G, it always follows that $w(G)$ is finite. More generally, a word w is said to be concise in a class of groups X if whenever G_w is finite for a group $G \in X$, it always follows that $w(G)$ is finite. P. Hall asked whether every word is concise, but later Ivanov proved that this problem has a negative solution in its general form [5] (see also [10, p. 439]). On the other hand, many relevant words are known to be concise. For instance, it was shown in [14] that the multilinear commutator words are concise. Such words are also known under the name of outer commutator words and are precisely the words that can be written in the form of multilinear Lie monomials. Merzlyakov showed that every word is concise in the class of linear groups [9] while Turner-Smith proved that every word is concise in the class of residually finite groups all of whose quotients are again residually finite [13]. There is an open problem whether every word is concise in the class of residually finite groups (cf. Segal [12, p. 15] or Jaikin-Zapirain [6]). It was shown in [1] that if w is a multilinear commutator word and n is a prime-power, then the word w^n is concise in the class of residually finite groups.

We say that a word w is boundedly concise in a class of groups X if for every integer m there exists a number $\nu = \nu(X, w, m)$ such that whenever $|G_w| \leq m$ for a group $G \in X$ it always follows that $|w(G)| \leq \nu$. Fernández-Alcober and Morigi [2] showed that every word which is concise in the class of all groups is actually boundedly concise. Moreover they showed that whenever w is a multilinear commutator word having at most m values in a group G, one has $|w(G)| \leq (m-1)^{(m-1)}$. It was shown in [1] that if $w = \gamma_k$ is the kth lower central word and n a prime-power, then the word w^n is boundedly concise in the class of residually finite groups. Recall that the word γ_k is defined inductively by the formulae

$$\gamma_1 = x_1, \qquad \gamma_k = [\gamma_{k-1}, x_k] = [x_1, \ldots, x_k], \quad \text{for } k \geq 2.$$

The corresponding verbal subgroup $\gamma_k(G)$ is the familiar kth term of the lower central series of G.

We say that w is *weakly rational* if for every finite group G and for every integer e relatively prime to $|G|$, the set G_w is closed under eth powers (note that the eth power map is a bijection on G). We say that w is *rational* if the number of solutions to the equation $w(x_1, \ldots, x_k) = g$ is the same as the number of solutions to $w(x_1, \ldots, x_k) = g^e$ for every e relatively prime to $|G|$. Clearly rational implies weakly rational.

The present article grew out of the observation that if w is a weakly rational word and G is a residually finite group in which w has at most m values, then the order of $w(G)$ is m-bounded.

Lemma 1. *The word w is weakly rational if and only if for every finite group G with $g \in G_w$, $g^e \in G_w$ whenever e relatively prime to $|g|$. The word w is rational if and only*