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Noncommutative complete intersections

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Several generalizations of a commutative ring that is a graded complete intersection are proposed for a noncommutative graded k-algebra; these notions are justified by examples from noncommutative invariant theory.

0. Introduction

Bass has noted that Gorenstein rings are ubiquitous [4]. Since the class of Gorenstein rings contains a wide variety of rings, it has proven useful to consider a tractable class of Gorenstein rings, and complete intersections fill that role for commutative rings.

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Similarly Artin–Schelter Gorenstein algebras, which are noncommutative generalizations of commutative Gorenstein rings, include a diverse collection of algebras, and finding a class of Artin–Schelter Gorenstein algebras that generalizes the class of commutative complete intersections is an open problem in noncommutative algebra. In this paper we use our work in noncommutative invariant theory to propose several notions of a noncommutative graded complete intersection. Moreover, the existence of noncommutative analogues of commutative complete intersection invariant subalgebras broadens our continuing project of establishing an invariant theory for finite groups acting on Artin–Schelter regular algebras that is parallel to classical invariant theory (see [28–32]).

When a finite group acts linearly on a commutative polynomial ring, the invariant subring is rarely a regular ring (the group must be a reflection group [45]), but Gorenstein rings of invariants are easily produced. For example, Watanabe's Theorem ([54] or [5, Theorem 4.6.2]) states that the invariant subring of $k[x_1, \cdots, x_n]$ under the natural action of a finite subgroup of $SL_n(k)$ is always Gorenstein, where k is a base field. In previous work we have shown there is a rich invariant theory for finite group (and even Hopf) actions on Artin–Schelter regular [Definition 3.1] (or AS regular, for short) algebras; for example there is a noncommutative version of Watanabe's Theorem [26, Theorem 3.3], providing conditions when the invariant subring is AS Gorenstein.

Cassidy and Vancliff defined a factor ring S/I of $S = k_{(q_{ij})}[x_1, \ldots, x_n]$, a skew polynomial ring, to be a complete intersection if I is generated by a regular sequence of length n in S (hence S/I is a finite dimensional algebra) [15, Definition 3.7], and in [52] Vancliff considered extending this definition to graded skew Clifford algebras. A few examples of noncommutative (or quantum) complete intersections have been constructed and studied along the line of factoring out a regular sequence of elements [9,10,42]. Further, different kinds of generalizations of a commutative complete intersection have been proposed during the last fifteen years [20,17,6,23]. Recent work on noncommutative (or twisted) matrix factorizations [13], derived representation schemes [8], noncommutative versions of support varieties and finite generation of the cohomology ring of a Hopf algebra (ideas similar to [7,41]), as well as noncommutative crepant resolutions of commutative schemes [16], advocate for a better understanding of noncommutative complete intersections. A satisfactory definition will have positive impact on several research areas.

In the commutative graded case a connected graded algebra A is called a *complete intersection* if one of the following four equivalent conditions holds [Lemma 1.7]

- (cci′) $A \cong k[x_1, \ldots, x_d]/(\Omega_1, \cdots, \Omega_n)$, where $\{\Omega_1, \ldots, \Omega_n\}$ is a regular sequence of homogeneous elements in $k[x_1, \ldots, x_d]$ with $\deg x_i > 0$.
- (cci) $A \cong C/(\Omega_1, \cdots, \Omega_n)$, where C is a noetherian AS regular algebra and $\{\Omega_1, \ldots, \Omega_n\}$ is a regular sequence of normalizing homogeneous elements in C.
- (gci) The Ext-algebra $E(A) := \bigoplus_{n=0}^{\infty} \operatorname{Ext}^n_A(k, k)$ of A has finite Gelfand–Kirillov dimension.
- (nci) The Ext-algebra $E(A)$ is noetherian.