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Partial monoid actions and a class of restriction semigroups

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We study classes of proper restriction semigroups determined by properties of partial actions underlying them. These properties include strongness, antistrongness, being defined by a homomorphism, being an action etc. Of particular interest is the class determined by homomorphisms, primarily because we observe that its elements, while being close to semidirect products, serve as mediators between general restriction semigroups and semidirect products or W-products in an embedding-covering construction. It is remarkable that this class does not have an adequate analogue if specialized to inverse semigroups. F-restriction monoids of this class, called ultra F-restriction monoids, are determined by homomorphisms from a monoid T to the Munn monoid of a semilattice Y. We show that these are precisely the monoids $Y *_m T$ considered by Fountain, Gomes and Gould. We obtain a McAlister-type presentation for the class given by strong dual prehomomorphisms and apply it to construct an embedding of ultra F-restriction monoids, for which the base monoid T is free, into W-products of semilattices by monoids. Our approach yields new and simpler proofs of two recent embedding-covering results by Szendrei.

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1. Introduction

Restriction semigroups, also known as weakly E-ample semigroups, are non-regular generalizations of inverse semigroups. These are semigroups with two additional unary operations which mimic the operations $a \mapsto a^{-1}a$ and $a \mapsto aa^{-1}$ on an inverse semigroup. Various aspects of restriction semigroups and their one-sided analogues have been extensively studied in the literature, see, e.g., [3,4,6,10] and the references therein.

Proper restriction semigroups are analogues of E-unitary inverse semigroups which play a central role in the theory of inverse semigroups and its applications. Generalizing corresponding results for inverse and ample semigroups [20,13,14], Cornock and Gould [1] gave a structure theorem for proper restriction semigroups in terms of double partial actions of monoids on semilattices. This can be readily reformulated in terms of only one partial action, since each of the two partial actions is determined by the other one. In the present paper we consider classes of proper restriction semigroups determined by the properties of this partial action. We show that the partial action is strong or antistrong if and only if the restriction semigroup satisfies a technical condition arising in [1]. W-products of semilattices by monoids correspond to the situation where the partial action is an action, and semidirect products form their subclass corresponding to actions by automorphisms. More importantly, we single out a class of proper restriction semigroups determined by homomorphisms. We call elements of this class ultra proper restriction semigroups. This is a rich and important class, since its elements, while being close to semidirect products, arise as mediators between general restriction semigroups and W-products or semidirect products in an embedding-covering result. It is interesting that this class does not have an adequate analogue if specialized to the inverse case, ultra proper inverse semigroups being precisely semidirect products of semilattices by groups. This discrepancy between restriction and inverse semigroups is well illustrated by the fact that free restriction monoids and semigroups are ultra proper, whereas free inverse monoids and semigroups are not. The subclass of ultra proper restriction semigroups which are also F-restriction (where F-restriction has a similar meaning as F-inverse in the inverse semigroup theory) is shown to be equal to the class of monoids $Y *_m T$ introduced by Fountain, Gomes and Gould in [4]. Alternatively, this subclass can be described by the property that the underlying homomorphism has its range in the Munn monoid of the semilattice. We call elements of this subclass ultra F-restriction monoids.

Based on ideas from [19] and [18], we construct a globalization of a strong partial action underlying a proper restriction semigroup and obtain a McAlister-type theorem. We then specialize this construction to partial actions defining ultra F-restriction monoids $M(T, Y)$, where the base monoid T is free. As a result, we obtain a semilattice X and an action of T on this semilattice such that the W-product $W(T, X)$ can be formed and, moreover, the initial monoid $M(T, Y)$ embeds into $W(T, X)$. This construction is inspired by Szendrei's embedding of the free restriction monoid into a W-product [21] and generalizes it. We apply the constructed embedding in the following setting. Let S