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c-Sections of Lie algebras

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ABSTRACT

Let M be a maximal subalgebra of a Lie algebra L and A/B a chief factor of L such that $B \subseteq M$ and $A \nsubseteq M$. We call the factor algebra $M \cap A/B$ a c-section of M. All such c-sections are isomorphic, and this concept is related those of c-ideals and ideal index previously introduced by the author. Properties of c-sections are studied and some new characterizations of solvable Lie algebras are obtained.

1. Preliminary results

Throughout L will denote a finite-dimensional Lie algebra over a field F. We denote algebra direct sums by '$\oplus$', whereas vector space direct sums will be denoted by '$\dot{+}$'. If B is a subalgebra of L we define B_L, the *core* (with respect to L) of B to be the largest

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ideal of L contained in B. In [10] we defined a subalgebra B of L to be a *c-ideal* of L if there is an ideal C of L such that $L = B + C$ and $B \cap C \subseteq B_L$.

Let M be a maximal subalgebra of L. We say that a chief factor C/D of L *supplements* M in L if $L = C + M$ and $D \subseteq C \cap M$; if $D = C \cap M$ we say that C/D *complements* M in L. In [11] we defined the *ideal index* of a maximal subalgebra M of L, denoted by $\eta(L : M)$, to be the well-defined dimension of a chief factor C/D where C is an ideal minimal with respect to supplementing M in L. Here we introduce a further concept which is related to the previous two.

Let M be a maximal subalgebra of L and let C/D be a chief factor of L with $D \subseteq M$ and $L = M + C$. Then $(M \cap C)/D$ is called a *c-section* of M in L. The analogous concept for groups was introduced by Wang and Shirong in [14] and studied further by Li and Shi in [3].

We say that L is *primitive* if it has a maximal subalgebra M with $M_L = 0$. First we show that all c-sections of M are isomorphic.

Lemma 1.1. *For every maximal subalgebra M of L there is a unique c-section up to isomorphism.*

Proof. Clearly c-sections exist. Let $(M \cap C)/D$ be a c-section of M in L, where C/D is a chief factor of L, $D \subseteq M$ and $L = M + C$. First we show that this c-section is isomorphic to one in which $D = M_L$. Clearly $D \subseteq M_L \cap C \subseteq C$, so either $M_L \cap C = C$ or $M_L \cap C = D$. If the former holds, then $C \subseteq M_L$, giving $L = M$, a contradiction. In the latter case put $E = C + M_L$. Then $E/M_L \cong C/D$ is a chief factor and $(M \cap E)/M_L$ is a c-section. Moreover,

$$\frac{M \cap E}{M_L} = \frac{M_L + M \cap C}{M_L} \cong \frac{M \cap C}{M_L \cap C} = \frac{M \cap C}{D}.$$

So suppose that $(M \cap C_1)/M_L$ and $(M \cap C_2)/M_L$ are two c-sections, where C_1/M_L, C_2/M_L are chief factors and $L = M + C_1 = M + C_2$. Then L/M_L is primitive and so either $C_1 = C_2$ or else $C_1/M_L \cong C_2/M_L$ and $C_1 \cap M = M_L = C_2 \cap M$, by [13, Theorem 1.1]. In the latter case both c-sections are trivial. □

Given a Lie algebra L with a maximal subalgebra M we define $Sec(M)$ to be the Lie algebra which is isomorphic to any c-section of M; we call the natural number $\eta^*(L : M) = \dim Sec(M)$ the *c-index* of M in L.

The relationship between c-ideals and c-sections, and between ideal index and c-index, for a maximal subalgebra M of L is given by the following lemma.

Lemma 1.2. *Let M be a maximal subalgebra of a Lie algebra L. Then*

(i) M is a c-ideal of L if and only if $Sec(M) = 0$; and
(ii) $\eta^*(L : M) = \eta(L : M) - \dim(L/M)$.