Journal of Algebra 427 (2015) 142–182

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

# Partial cohomology of groups

M. Dokuchaev [1], M. Khrypchenko [*,2]

*Instituto de Matemática e Estatística, Universidade de São Paulo, Caixa Postal 66281, São Paulo, SP 05315-970, Brazil*

A R T I C L E   I N F O

*Article history:*
Received 20 February 2014
Available online 14 January 2015
Communicated by Volodymyr Mazorchuk

*MSC:*
primary 20J06
secondary 18G60, 20C25, 20M18, 20M30, 20M50

*Keywords:*
Partial action
Partial Schur multiplier
Partial group cohomology
Cohomology of inverse semigroups

A B S T R A C T

We develop a cohomology theory of groups based on partial actions and explore its relation with the partial Schur multiplier as well as with cohomology of inverse semigroups.


## Introduction

In [15,17,18] R. Exel initiated a successful method to study $C^*$-algebras, which is based on new concepts, namely, those of a partial action, the corresponding crossed product and a partial representation, as well as on the interaction between them. Relevant classes of $C^*$-algebras have been shown to have the structure of a non-trivial crossed product by

* Corresponding author.
*E-mail addresses:* dokucha@gmail.com (M. Dokuchaev), nskhripchenko@gmail.com (M. Khrypchenko).
1 Partially supported by CNPq of Brazil (process number: 305975/2013-7).
2 Supported by FAPESP of Brazil (process number: 2012/01554-7).

http://dx.doi.org/10.1016/j.jalgebra.2014.11.030
0021-8693/

a partial action, permitting one to investigate their internal structure, representations and K-theory (see [14,16,20,33,1]).

The first algebraic results on the above mentioned notions were established in [18,7,35,25,6], which together with the development of the partial Galois theory in [10] stimulated an intensive algebraic activity on partial actions, corresponding crossed products and partial representations (see the surveys [5] and [22]). In particular, applications were obtained to graded algebras in [6] and [8], to Hecke algebras in [19] and to Leavitt path algebras in [23].

A purely ring theoretic version of the general notion of a twisted partial group action, initially introduced in the context of $C^*$-algebras in [17], was given in [8]. It allowed one to show that any group graded algebra, satisfying some reasonably mild restrictions, is stably isomorphic (in a certain sense) to a crossed product by a twisted partial action, establishing thus an algebraic counterpart of a result from [17]. The concept involves a general twisting which satisfies the 2-cocycle identity in some restricted sense, and one naturally wonders what kind of cohomology theory would fit this. It is complicated to give such a theory which would embrace the full generality of [8], since the twisting takes its values in multiplier algebras of products of some ideals indexed by group elements. Nevertheless, imposing a reasonable restriction on the partial action, namely, that it is unital, makes it possible to define partial cohomology. Actually, this restriction is assumed in almost all algebraic papers around partial actions, because it provides an appropriate technical framework.

One feels more confident in developing such a cohomology theory if one shows that it matches some other concepts in a way how the usual group cohomology does. In particular, the Schur multiplier of a group $G$ over the complex numbers $\mathbb{C}$ is isomorphic to $H^2(G, \mathbb{C}^*)$ with trivial action of $G$ on $\mathbb{C}^*$. Thus a theory of partial projective group representations could be a testing field for our cohomology. Such a theory was developed in [11–13], in particular the structure of the partial Schur multiplier was explored. Further results on the latter topic were obtained in [31] and [32].

A twisted partial action of a group $G$ on a commutative ring $A$ falls into two parts: a partial action $\theta$ of $G$ on $A$ and its twisting. With an additional assumption that $\theta$ is unital (which means that the domains involved in $\theta$ are generated by central idempotents), one can derive the concept of a partial 2-cocycle (the twisting) whose values belong to groups of invertible elements of appropriate ideals of $A$. The concept of a partial 2-coboundary then follows from that of an equivalence of twisted partial actions introduced in [9]. Replacing $A$ by a commutative multiplicative monoid, one comes to the definition of the second cohomology group $H^2(G, A)$. Thus instead of a usual $G$-module we deal with a partial $G$-module, which is a commutative monoid $A$ with a unital partial action $\theta$ of $G$ on $A$. The groups $H^n(G, A)$ with arbitrary $n$ are defined in a similar way (see Section 1). Replacing $A$ by an appropriate submonoid one may actually assume that $A$ is inverse (see Remark 3.14). Next one asks how to obtain these groups using, say, projective resolutions. It turns out that the category of partial $G$-modules is not abelian since some sets of morphisms may be empty. Fortunately, our cohomology can be related