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$\mathbb{Z}$-graded identities of the Lie algebra W_1

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ABSTRACT

Let K be a field of characteristic 0 and let W_1 be the Lie algebra of the derivations of the polynomial ring $K[t]$. The algebra W_1 admits a natural $\mathbb{Z}$-grading. We describe the graded identities of W_1 for this grading. It turns out that all these $\mathbb{Z}$-graded identities are consequences of a collection of polynomials of degree 1, 2 and 3 and that they do not admit a finite basis. Recall that the "ordinary" (non-graded) identities of W_1 coincide with the identities of the Lie algebra of the vector fields on the line and it is a long-standing open problem to find a basis for these identities. We hope that our paper might be a step to solving this problem.

1. Introduction

The development of the structure theory for the T-ideals in the free associative algebra was one of the major achievements in the PI theory. This development is mainly asso-

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ciated with the results obtained by Kemer in the 1980-ies. Details concerning Kemer's fundamental contributions to this theory can be found in several monographs [9,7,8]. The key points in Kemer's theory are the description of the T-ideals in terms of identities satisfied by certain "good" algebras: finitely generated and finite dimensional ones, as well as the connections between ordinary and graded identities. Soon after Kemer's work graded algebras and graded polynomial identities became objects of independent interest. The reader may find an extensive list of references concerning recent results about gradings on algebras and their graded identities in [3].

It is somewhat surprising that graded identities for Lie algebras have not been studied to that extent. (This is not the case with Lie superalgebras where the proper defining relations involve gradings.) The group gradings for many finite dimensional simple Lie algebras over an algebraically closed field K of characteristic $\neq 2$ were classified in [2], see also [4]. However, the only nontrivial instance where we know exactly the graded identities is that of $sl_2(K)$ whenever K is an infinite field and $\operatorname{char} K \neq 2$. The relatively free algebras corresponding to the graded identities satisfied by $sl_2(K)$ endowed with each of the three possible nontrivial gradings were described in [15]. Bases of these graded identities were exhibited in [10] (it should be mentioned that the results in [10] hold for K infinite, $\operatorname{char} K \neq 2$), see a streamlined and simplified version in [11]. Recently, it was proved in [5] that over a filed K of characteristic 0 the T-ideal of graded identities of $sl_2(K)$ has the Specht property. The growth of the variety of graded Lie algebras generated by $sl_2(K)$ was found in [6].

In [12] it was proved that over an infinite field K of characteristic 2 the $\mathbb{Z}_2$-graded identities of the Lie algebra $gl_2(K)$ do not admit any finite basis. Moreover an example of a just nonfinitely based variety of $\mathbb{Z}_2$-graded algebras was constructed in [12]. (Recall that a just nonfinitely based variety is a variety without finite basis of its identities whose proper subvarieties admit such a finite basis.)

Let K be a fixed field of characteristic 0 and let G be a group. An algebra (not necessarily associative) A over K is G-graded if $A = \bigoplus_{g\in G} A_g$, a direct sum of vector subspaces such that $A_g A_h \subseteq A_{gh}$ for every g, $h \in G$. If $G = \mathbb{Z}$ is the additive group of the integers then A is $\mathbb{Z}$-graded. In this case $A_g A_h \subseteq A_{g+h}$.

Now consider the polynomial algebra $K[t]$ in one variable t. The derivations of the algebra $K[t]$ form a Lie algebra denoted by W_1, $W_1 = \operatorname{Der}(K[t])$. It is immediate that the elements $e_n = t^{n+1}d/dt$, $n \geq -1$, form a basis of W_1. The Lie algebra structure on the vector space W_1 is given by the multiplication

$$[e_i, e_j] = (j - i)e_{i+j}. \tag{1}$$

The algebra W_1 is $\mathbb{Z}$-graded, $W_1 = \bigoplus_{i\in\mathbb{Z}} L_i$ where $L_i = 0$ whenever $i \leq -2$, and L_i is the span of $t^{i+1}d/dt$ if $i \geq -1$. The algebra W_1 is known as the *Witt algebra* and plays an important role in the applications.

One denotes by W_n the Lie algebra of derivations of the polynomial algebra $K[x_1, \ldots, x_n]$, $n \geq 1$. It is well known that the Lie algebras W_n form one of the se-