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Minimal characteristic bisets for fusion systems

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ABSTRACT

We show that every saturated fusion system $\mathcal{F}$ has a unique minimal $\mathcal{F}$-characteristic biset $\Lambda_{\mathcal{F}}$. We examine the relationship of $\Lambda_{\mathcal{F}}$ with other concepts in p-local finite group theory: In the case of a constrained fusion system, the model for the fusion system is the minimal $\mathcal{F}$-characteristic biset, and more generally, any centric linking system can be identified with the $\mathcal{F}$-centric part of $\Lambda_{\mathcal{F}}$ as bisets. We explore the grouplike properties of $\Lambda_{\mathcal{F}}$, and conjecture an identification of normalizer subsystems of $\mathcal{F}$ with subbisets of $\Lambda_{\mathcal{F}}$.

1. Introduction

If S is a Sylow p-subgroup of a finite group G, we talk about the fusion system $\mathcal{F}_S(G)$ as an organizational framework for understanding the p-local structure of G. The fusion data is encoded as a category: The objects of $\mathcal{F}_S(G)$ are the subgroups of S, and the morphisms are the maps between subgroups induced by conjugation in G. More generally,

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Puig introduced the notion of an abstract fusion system on S: This is again a category $\mathcal{F}$ with objects the subgroups of S and morphisms certain injective group maps between subgroups (see Section 2).

An abstract fusion system does not necessarily arise from a group in this manner, but we still think of the morphisms in $\mathcal{F}$ as given by the conjugation action of some grouplike object on the subgroups of S. The notion of a *characteristic biset* turns this perspective around, and considers how S acts on the object that does the conjugating.

For $S \in \mathrm{Syl}_p(G)$ and the fusion system $\mathcal{F}_S(G)$ induced by G-conjugation on S, we can ask how S acts on G by left and right multiplication. That is, we consider the (S,S)-biset ${}_SG_S$. For any $g \in G$ and any pair $(b,a) \in S \times S$, asking that $b \cdot g = g \cdot a$ is satisfied in the biset ${}_SG_S$, is equivalent to $b = {}^ga$. In other words, fusion data ($b = {}^ga$) is encoded in the biset structure ($b \cdot g = g \cdot a$). This justifies calling ${}_SG_S$ a *characteristic biset* for $\mathcal{F}_S(G)$.

Linckelmann and Webb extracted the features of ${}_SG_S$ that are essential for understanding the fusion system $\mathcal{F}_S(G)$, resulting in a notion of characteristic bisets for any abstract fusion system $\mathcal{F}$. Fix a p-group S, a fusion system $\mathcal{F}$ on S, and an (S,S)-biset Ω. Ω is then a characteristic biset for $\mathcal{F}$ if:

(0) Ω is free both as a left and right S-set.

This means that any $\omega \in \Omega$ has stabilizer $\{(b,a) \in S \times S \mid b \cdot \omega = \omega \cdot a\}$ of the form $(P,\varphi) := \{(\varphi(a),a) \mid a \in P\}$ for P a subgroup of S and $\varphi\colon P \hookrightarrow S$ some group injection.
Heuristically, this says that ω “conjugates” a to $\varphi(a)$.

(1) If $\omega \in \Omega$ has stabilizer (P,φ), then φ is a morphism of $\mathcal{F}$.

This means that all the conjugation induced by Ω is in $\mathcal{F}$.

(2) For subgroups P of S, $\mathcal{F}$-morphisms $\varphi\colon P \to S$, and $\mathcal{F}$-isomorphisms $\eta_1\colon Q \xrightarrow{\cong} P$, $\eta_2\colon \varphi P \xrightarrow{\cong} R$, there is an equality of fixed-point set orders: $|\Omega^{(P,\varphi)}| = |\Omega^{(Q,\eta_2\varphi\eta_1)}|$.

This condition generalizes the fact that, if G acts on a set X, then conjugate subgroups of G have fixed-point sets of equal size.

(3) $|\Omega|/|S|$ is prime to p.

This Sylow condition generalizes $S \in \mathrm{Syl}_p(G)$.

The connection between a fusion system $\mathcal{F}$ and an associated $\mathcal{F}$-characteristic biset is very strong:

- If $\mathcal{F}$ is *saturated* (i.e., it satisfies the axioms needed to make $\mathcal{F}$ look like the fusion induced by a finite group), then there exists a characteristic biset [3].
- If a characteristic biset for $\mathcal{F}$ exists, then $\mathcal{F}$ is saturated ([11], also see [14] for a p-localized version).
- As suggested by Axioms (1) and (2), the characteristic biset determines $\mathcal{F}$.