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GKM theory for p-compact groups

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ABSTRACT

This work studies the flag varieties of p-compact groups, principally through torus-equivariant cohomology, extending methods and tools of classical Schubert calculus and moment graph theory from the setting of real reflection groups to the broader context of complex reflection groups. In particular we give, for the infinite family of p-compact flag varieties corresponding to the complex reflection groups $G(r, 1, n)$, a generalized GKM characterization (following Goresky–Kottwitz–MacPherson [8]) of the torus-equivariant cohomology, building an explicit additive basis and showing its relationship with the polynomial or Borel presentation via the localization map.

1. Introduction

p-Compact groups are the culmination of an endeavour to extract the homotopical properties of compact Lie groups that started with the work of Hopf [10] and Serre [15] on H-spaces and loop spaces. p-Compact groups are objects defined solely in homotopy theoretic terms that resemble compact Lie groups in many ways. For instance, p-compact groups have Weyl groups that classify them and encode plenty of information about them,

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just as in the Lie group case. They also have maximal Tori, with which one can build p-compact flag varieties.

Our object of study is precisely the p-compact flag variety, which, contrary to the compact Lie group case, is not necessarily a manifold or variety: the structure of p-compact flag varieties is purely homotopical.

In this work we show how the torus-equivariant cohomology of p-compact flag varieties is described in terms of the Weyl group. The Weyl groups of p-compact groups are $\mathbb{Z}_p$-reflection groups, which can be thought of as $\mathbb{C}$-reflection groups. Moreover, finite reflection groups can be classified as Weyl groups of p-compact groups, as shown in [1].

A concise summary of the fundamental results in the theory of p-compact groups can be found in [13].

In a very broad sense one can say that Schubert calculus is the study of the cohomology of flag varieties. Inside Schubert calculus, GKM theory deals with the torus-equivariant cohomology of flag varieties making use of moment (or Bruhat) graphs. In this paper we intersect GKM theory with the theory of p-compact groups to study the torus-equivariant cohomology of p-compact flag varieties within the more general framework of equivariant homotopy theory.

This article is organized as follows: In Section 2 we will define torus-equivariant cohomology for p-compact flag varieties and review the Borel (or polynomial) and Schubert presentations of this cohomology. In Section 3 we introduce the concept of moment graph and its associated structure algebra as the means to describe the GKM theory of p-compact flag varieties. Sections 4 and 5 will establish the connection between the moment graph and the cohomology of flag varieties, and its relation to the Borel presentation via the localization map. Here we will build an explicit additive basis for the equivariant cohomology in type $G(r, 1, n)$, both in the moment graph and polynomial presentations. Finally in Section 6 we will state our main result, the GKM presentation of p-compact flag varieties corresponding to the complex reflection groups $G(r, 1, n)$.

Another important consequence of our results is to confirm that, for the groups $G(r, 1, n)$, the structure algebra is a free module over the symmetric algebra, by constructing an explicit basis. In general, the structure algebra associated to an arbitrary reflection group need not be a free module, though we expect it to be so from what is known in the classical case.

2. Borel presentation and Schubert presentation

Let X be a p-compact group with maximal torus T. The **flag variety** of X is the homotopy fiber of the map $BT \longrightarrow BX$. It is denoted by X/T.

Define $BT \times_{BX} BT$ as the fiber product of the fibration $X/T \hookrightarrow BT \longrightarrow BX$ with itself, i.e. the pullback of the following diagram