Journal of Algebra 420 (2014) 15–38

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

# The clean property is not a Morita invariant

Janez Šter

*Department of Mathematics, University of Ljubljana, Slovenia*

A R T I C L E I N F O

*Article history:*
Received 22 October 2013
Available online 6 September 2014
Communicated by Louis Rowen

*MSC:*
16U99
16E20

*Keywords:*
Clean ring
Morita invariant
Matrix ring
Bergman's example
$K$-theory

A B S T R A C T

For each $n \geq 2$, we construct a ring $R$ such that the matrix ring $M_n(R)$ is clean and $M_k(R)$ is not clean for $k < n$. This answers in negative the question posed by Han and Nicholson [11] whether the clean property is Morita invariant.



## 1. Introduction

The notion of a clean ring was introduced in 1977 by Nicholson [15], as a ring in which every element can be expressed as the sum of an idempotent and a unit. Nicholson proved that clean rings are a subclass of exchange rings, and that the converse also holds in the case when idempotents in the ring are central.

In the past, the study of clean rings was mostly focused on verifying which known exchange rings are clean. Thus, it was proved that semiperfect rings [7], unit-regular rings [5], and endomorphism rings of vector spaces [16] are all clean. In addition, endomorphism rings of continuous modules [6] and exchange rings with primitive factors

http://dx.doi.org/10.1016/j.jalgebra.2014.07.030

artinian [8,13] are clean. These examples show that clean rings are quite a large subclass of exchange rings.

The first example of a non-clean exchange ring was found by Camillo and Yu [7] in 1994. Camillo and Yu observed that Bergman's example [12, Example 1], of a regular ring that is not generated by its units, is not clean. Therefore, this example proved that regular rings (which are always exchange) need not be clean.

One advantage of clean rings over exchange rings is that the group $K_0$ of clean rings with a "large" ideal behaves nicer than in the case of exchange rings. In particular, in [21] it was proved that, if $R$ is a clean ring with an ideal $I$ such that $R/I$ is local, then units in $R/I$ can be lifted modulo $I$ and we have $K_0(R) \cong K_0(R/I) \oplus K_0(I)$. Also, if $R$ is clean, $R/I$ is semiperfect and $K_0(I)$ is torsion-free, then, under mild conditions, units lift modulo $I$ and we have $K_0(R) \cong K_0(R/I) \oplus K_0(I)$. These properties can easily fail in exchange rings, as Bergman's example demonstrates (see [21]).

In 2001, Han and Nicholson [11] proved that clean rings are closed under matrix extensions. More generally, if $e$ is an idempotent in a ring $R$ such that the corner rings $eRe$ and $(1-e)R(1-e)$ are clean, then $R$ is also clean. This implies that the matrix ring $M_n(R)$ over a clean ring $R$ is clean, for every $n$. Han and Nicholson also asked if the converse of this proposition holds: if $M_n(R)$ is clean, is then $R$ also clean. More generally, they asked if the clean property is a Morita invariant, i.e. if $eRe$ is a clean ring whenever $R$ is clean and $e$ is a *full* idempotent in $R$ (an idempotent $e \in R$ with $R = ReR$).

In [20] an example was given showing that, in general, corners of clean rings need not be clean. However, the example given there did not answer the question about the Morita invariance since the corner rings in that example were not full. The main purpose of this paper is to give an example of a ring such that the matrix ring is clean but the ring itself is not. In fact, for every $n$ we give an example of a ring $R$ such that $M_k(R)$ is clean if and only if $k$ is a multiple of $n$. These examples therefore show that the clean property is not Morita invariant.

The idea behind our construction comes from the above mentioned $K$-theoretic property of clean rings. In particular, we can find a clean ring $R$ with an ideal $I$ such that $R/I$ is semiperfect and $K_0(I)$ has torsion, but units do not lift modulo $I$. This indeed turns out to be the key idea to finding the example.

The paper is organized as follows. In Section 2 we give some preliminary results, and introduce terminology used throughout the paper. In Section 3 we construct an example of a clean ring $R$ such that the group $K_0(R)$ is cyclic of order $n$, generated by the module $R_R$. This ring is obtained from Goodearl's example of a regular ring with the group $K_0$ cyclic of order $n$.

In Section 4 we define a generalization of Bergman's example of a non-clean exchange ring. We also prove some basic properties of the obtained ring. In Section 5 we prove that any non-scalar matrix (in a suitable sense) over generalized Bergman's ring is a sum of an idempotent and a unit. In the last section we put together all the previous results to obtain the desired example.