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On the outer automorphism groups of free groups

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A R T I C L E I N F O

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We prove that the outer automorphism group of a free group of countably infinite rank is complete.

Introduction

Khramtsov [12] proved in 1990 that the outer automorphism group $\mathrm{Out}(F_n)$ of a free group F_n of finite rank $n \geqslant 3$ is complete; his proof used the results on the structure of finite subgroups of the group $\mathrm{Out}(F_n)$. Later, in 2000, Bridson and Vogtmann [1] obtained another proof of completeness of the group $\mathrm{Out}(F_n)$ based on the results on the action of the group $\mathrm{Out}(F_n)$ on the Culler–Vogtmann Outer Space.

The quoted results on the outer automorphism groups $\mathrm{Out}(F_n)$ have been the natural development of the earlier results by Dyer and Formanek [6–8] on the automorphism groups of relatively free groups of finite rank. One of the main results obtained by Dyer

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and Formanek [6] stated that the automorphism group $\mathrm{Aut}(F_n)$ is complete for all $n \geqslant 2$. In 2000, the author extended this result to arbitrary free nonabelian groups by showing that the automorphism group of any infinitely generated free group is also complete [14].

The main result of the present paper states completeness of the outer automorphism group of a free group of countably infinite rank. The similar result on free groups of arbitrary infinite rank seems to be also true, but the main tool we rely upon in our paper—the small index property for relatively free algebras—is established only for free groups of countably infinite rank. We say that a relatively free algebra $\mathfrak{F}$ of infinite rank $\varkappa$ has the *small index property*, if any subgroup Σ of the automorphism group $\Gamma = \mathrm{Aut}(\mathfrak{F})$ of index at most $\varkappa$ contains the pointwise stabilizer $\Gamma_{(U)}$ of a subset U of $\mathfrak{F}$ of cardinality $< \varkappa$ [16] (in the case when $\mathfrak{F}$ is countable, this property coincides with the small index property for *countable* structures, much studied since the early 1980s). As it has been demonstrated in [16], the small index property is true for arbitrary infinitely generated free nilpotent groups. The case of infinitely generated free groups appears to be much harder, the validity of the small index property being established by Bryant and Evans in [2] only for free groups of countably infinite rank.

Given a group G and a relation $R \subseteq G^n$ on G, we shall call R *definable* in G if it can be characterized in G in terms of group operation; any definable relation on G is invariant under all automorphisms of G. If a relation R on G admits a group-theoretic characterization in G involving relations $R_1^*, \dots, R_n^*$ on G, we shall say that R is definable in G with *parameters* $R_1^*, \dots, R_n^*$. Accordingly, if all parameters $R_1^*, \dots, R_n^*$ are invariant under a certain automorphism σ of G, then any relation R which is definable in G with parameters $R_1^*, \dots, R_n^*$ is also invariant under σ.

Let F denote a free group of infinite rank.

The paper is organized as follows. The bulk of the first two sections is devoted to definability of so-called extremal involutions in the group $\mathrm{Out}(F)$.

An involution $\varphi \in \mathrm{Aut}(G)$ where G is a relatively free group is said to be *extremal* if there is a basis $\mathscr{B}$ of G such that φ inverts some element of $\mathscr{B}$ and fixes all other elements of $\mathscr{B}$. Respectively, an involution $f \in \mathrm{Out}(G)$ is called *extremal* if it is induced (under the natural homomorphism $\mathrm{Aut}(G) \to \mathrm{Out}(G)$) by an extremal involution from $\mathrm{Aut}(G)$. In the first section we prove definability of the family of involutions from $\mathrm{Out}(F)$ which induce diagonalizable involutions in the automorphism group $\mathrm{Aut}(A)$ of the abelianization A of F.

In the second section we first prove definability of a certain family of 1-involutions in the group $\mathrm{Out}(F)$ (involutions that induce extremal involutions in the group $\mathrm{Aut}(A)$), and then work to obtain an explicit group-theoretic characterization of the extremal involutions in the group $\mathrm{Out}(F)$. Note that the extremal involutions are also used heavily in the paper [12] by Khramtsov, but he does not provide an explicit description of them.

Let $\mathscr{B}$ be a basis of F. In the third section, based on definability of extremal involutions, we prove that an arbitrary automorphism $\Delta \in \mathrm{Aut}(\mathrm{Out}(F))$ can be followed by a suitable inner automorphism to ensure that the resulting automorphism Δ' fixes pointwise the images of all $\mathscr{B}$-finitary automorphisms of the group F in the group $\mathrm{Out}(F)$.