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Bazzoni–Glaz conjecture

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In [2], Bazzoni and Glaz (2007) conjecture that the weak global dimension of a Gaussian ring is $0, 1$ or ∞. In this paper, we prove their conjecture.

1. Introduction

In her thesis [11], H. Tsang, a student of Kaplansky introduced Gaussian rings. Noting that the content of a polynomial f over a commutative ring R is the ideal $c(f)$ generated by the coefficients of f, we now define a Gaussian ring.

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Definition 1.1. (See Tsang, [11].) A polynomial $f \in R[x]$ is called Gaussian if $c(f)c(g) = c(fg)$ for all $g \in R[x]$. The ring R is called Gaussian if each polynomial in $R[x]$ is Gaussian.

In [2], the authors consider five possible extensions of the Prüfer domain notion to the case of commutative rings with zero divisors, two among which are Gaussian rings and rings with weak global dimension (see Definition 2.2) at most one. They consider the problem of determining the possible values for the weak global dimension of a Gaussian ring. At the end of their paper, they make the following conjecture.

Conjecture. *(See Bazzoni and Glaz, [2].) The weak global dimension of a Gaussian ring is either* $0, 1$ *or* ∞.

Our aim in this article is to give a proof of the above conjecture. The above conjecture is also listed as an open question in the recent survey article [6]. In a recent paper [1], the authors have validated the Bazzoni–Glaz conjecture for the class of rings called fqp-rings. The class of fqp-rings fall strictly between the classes of arithmetical rings and Gaussian rings. In [5], the author shows that the weak global dimension of a coherent Gaussian ring is either ∞ or at most one. She also shows that the weak global dimension of a Gaussian ring is at most one if and only if it is reduced. So to prove the conjecture, it is enough to show that $\text{w.gl.dim}\, R = \infty$ for all non-reduced Gaussian rings R. Since $\text{w.gl.dim}\, R = \sup\{\text{w.gl.dim}\, R_{\mathfrak{p}} \mid \mathfrak{p} \in \text{Spec}(R)\}$, it is enough to prove the conjecture for non-reduced local Gaussian rings. For any non-reduced local Gaussian ring R with nilradical $\mathcal{N}$, either (i) $\mathcal{N}$ is nilpotent or (ii) $\mathcal{N}$ is not nilpotent. Except when $\mathcal{N}^2 = 0$, the authors of [2] prove that if R satisfies (i), then $\text{w.gl.dim}\, R = \infty$. In this paper we prove that if R satisfies (ii), then $\text{w.gl.dim}\, R = \infty$ (cf. Theorem 5.4). We also give a complete proof of (i). Now we briefly describe the strategy of the proof. After localization at minimal prime ideal, $\mathcal{N}$, the maximal ideal and the nilradical of $R_{\mathcal{N}}$ coincide, let us denote it by $\mathcal{N}'$. If $\mathcal{N}^2 \neq 0$, then $\mathcal{N}' \neq 0$. Hence to prove the Bazzoni–Glaz conjecture in this case, it suffices to show that the weak global dimension of a local Gaussian ring with the maximal ideal coinciding with the nilradical is infinite. If $\mathcal{N}^2 = 0$, then $\mathcal{N}'$ can be zero (see Example 5.3). Hence we discuss this case separately in Section 6.

In Section 3, we consider some homological properties of local Gaussian rings. In particular we consider local Gaussian rings $(R, \mathfrak{m})$ which are not fields, with the property that each element of $\mathfrak{m}$ is a zero divisor. In this case we prove that $\text{w.gl.dim}\, R \geq 3$.

In [2, Section 6], the authors consider local Gaussian rings $(R, \mathfrak{m})$ such that the maximal ideal $\mathfrak{m}$ coincides with the nilradical of R. With this set up in Section 4, we prove that if $\mathcal{N}$ is not nilpotent, then $\text{w.gl.dim}\, R = \infty$.

In Section 5, we prove the conjecture except the case $\mathcal{N}^2 = 0$, and in the last section we prove it completely.

Throughout this paper, R is a commutative ring with unit, $(R, \mathfrak{m})$ is a local ring (not necessarily Noetherian) with unique maximal ideal $\mathfrak{m}$. For R-modules M and N, we write