Journal of Algebra 420 (2014) 161–185

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Equivalence and resolution of singularities

Augusto Nobile

Louisiana State University, Department of Mathematics, Baton Rouge, LA 70803, USA

A R T I C L E   I N F O

Article history:
Received 20 September 2013
Available online 6 September 2014
Communicated by Steven Dale Cutkosky

MSC:
14E15
13H15
14B05
14B99
14E99

Keywords:
Resolution of singularities
Multiplicity
Basic and Rees objects
Blowing up
Permissible centers
Equivalence

A B S T R A C T

This article provides a simple presentation of an algorithm to resolve singularities of algebraic varieties over fields of characteristic zero by means of a sequence of blowing ups with smooth centers contained in the set of points of maximum multiplicity. The algorithm uses primarily multiplicity, rather than the Hilbert–Samuel function, to control the resolution process, and it does not involve a local embedding into a smooth variety. The paper introduces a generalization of the usual notion of equivalence in the theory of resolution of singularities, which is important to justify an essential step in the construction of the algorithm.

Introduction

At present, following Hironaka's ideas (see, e.g., [14]) the "constructive" proofs of resolution of singularities of algebraic varieties over fields of characteristic zero proceed by proving first a more technical resolution theorem for some auxiliary objects (called

E-mail address: nobile@math.lsu.edu.

http://dx.doi.org/10.1016/j.jalgebra.2014.07.022

basic objects, marked ideals, presentations, etc., by different authors), and then showing that this result implies classical resolution of singularities. See [5,6,8,9,20,25], among others. The original proof is in [13]. The more promising work toward resolution in characteristic p obtained so far also involves a similar technique (e.g., [3,22,15]).

The best results concerning the passage from resolution of basic objects to that of algebraic varieties are usually gotten by using the Hilbert–Samuel (HS) function. This has two disadvantages. First, this is a rather complicated algebraic concept, which does not have a simple geometric interpretation.

Second, usually one controls the set of points M where the HS function of a variety X reaches its maximum value by *locally* embedding X into a smooth variety W, and introducing a basic object defined on W whose singular set (1.5) or cosupport [6] coincides with M. We may say that this basic object *represents* the HS function. There are several choices involved, so it is not clear that this locally defined method globalizes. One of the main problems is the fact that the mentioned local embedding can be done only locally (see [16]), and in many ways. See [1,4] for (rather complicated) proofs of the independence of the choices.

In a still unpublished recent paper [23] Villamayor proposed another approach to derive resolution of varieties from resolution of basic objects. The main tool of this method is the use of *multiplicity* rather than the HS function. And the "representation" of the set of points of maximal multiplicity of a variety X as the singular set of a basic object is not made via a local embedding, near a point x into a smooth variety W (of higher dimension) but rather by considering a smooth variety V, admitting a suitable *finite* morphism (or projection) $X' \to V$, where X' is an appropriate étale neighborhood of $x \in X$.

The articles [23,24] and [4] discuss the mentioned facts (or some closely related ones) in a rather general context, and they include a number of very interesting results, but some of them do not seem essential for a proof of the main resolution theorems that appear in those papers. It would be nice to have a presentation of those fundamental results on resolution as brief and direct as possible, and this is what the present paper intends to do.

Aside from the organization of the material, a difference of our approach is that to deal with an important gluing problem (to show that a certain locally defined process globalizes), we emphasize, following [6] and [18], the use of functorial arguments. In [18], this method involves the notion of *equivalence* (or, for some authors, weak equivalence). Essentially, basic objects B and B', defined over the same smooth variety, are equivalent when their singular loci coincide, and this equality is preserved when we apply to them certain transformations (2.1). Using the method of projections of [23], in the present situation one is naturally led to a situation where we'd like to have a similar concept, but for basic objects defined over different varieties. We introduce and use of the notion of what we call *P-equivalence* to formalize such an idea.