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Simplicity of partial skew group rings with applications to Leavitt path algebras and topological dynamics

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A B S T R A C T

Let R_0 be a commutative and associative ring (not necessarily unital), G a group and α a partial action of G on ideals of R_0, all of which have local units. We show that R_0 is maximal commutative in the partial skew group ring $R_0 \rtimes_\alpha G$ if and only if R_0 has the ideal intersection property in $R_0 \rtimes_\alpha G$. From this we derive a criterion for simplicity of $R_0 \rtimes_\alpha G$ in terms of maximal commutativity and G-simplicity of R_0. We also provide two applications of our main results. First, we give a new proof of the simplicity criterion for Leavitt path algebras, as well as a new proof of the Cuntz–Krieger uniqueness theorem. Secondly, we study topological dynamics arising from partial actions on clopen subsets of a compact set.

1. Introduction

Partial skew group rings arose as a generalization of skew group rings and as an algebraic analogue of C*-partial crossed products (see [5]). Much in the same way as

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skew group rings, partial skew group rings provide a way to construct non-commutative rings, and recently Leavitt path algebras have been realized as partial skew group rings (see [10]), indicating that the theory of non-commutative rings may benefit from the theory of partial skew group rings. Still, when compared to the well-established theory of skew group rings, the theory of partial skew group rings is still in its infancy. In fact, to our knowledge, [3] and [4] are the only existing papers regarding the ideal structure of partial skew group rings, and [9] is a recent paper describing simplicity conditions for partial skew group rings of abelian groups.

Our main goal in this paper is to derive necessary and sufficient conditions for simplicity of partial skew group rings. In general, this is still an open problem, even for skew group rings. In [12] and [14], Öinert has attacked this problem for skew group rings $R_0 \rtimes_\alpha G$, where either the group G, or the ring R_0, is abelian. Recently, in [9], a criterion for simplicity of partial skew group rings of abelian groups has been described. In our case, we will extend results of [12] to partial skew group rings $R_0 \rtimes_\alpha G$, where R_0 is assumed to be commutative and associative (not necessarily unital) and α is a partial action on ideals of R_0, all of which have local units. More specifically, we will show that $R_0 \rtimes_\alpha G$ is simple if and only if R_0 is G-simple and maximal commutative in $R_0 \rtimes_\alpha G$. In particular, our results can be applied to Leavitt path algebras, by realizing them as partial skew group rings (see [10]), and to partial skew group rings associated with partial topological dynamics.

Our work is organized in the following way: In Section 2 we present our main results, preceded by a quick overview of the key concepts involved below. In Section 3 we apply the results of Section 2 to derive a new proof of the simplicity criterion for Leavitt path algebras, as well as a new proof of the Cuntz–Krieger uniqueness theorem for Leavitt path algebras. In Section 4 we show an application of the results of Section 2 to partial topological dynamics, namely to partial actions by clopen subsets of a compact set.

Recall that a partial action of a group G (with identity element denoted by e) on a set Ω, is a pair $\alpha = (\{D_t\}_{t\in \mathrm{G}}, \{\alpha_t\}_{t\in \mathrm{G}})$, where for all $s, t \in \mathrm{G}$, D_t is a subset of Ω and $\alpha_t : D_{t^{-1}} \to D_t$ is a bijection such that $D_e = \Omega$, α_e is the identity map on Ω, $\alpha_t(D_{t^{-1}} \cap D_s) = D_t \cap D_{ts}$ and $\alpha_t(\alpha_s(x)) = \alpha_{ts}(x)$, for all $x \in D_{s^{-1}} \cap D_{s^{-1}t^{-1}}$. In case Ω is a ring (algebra) then, for each $t \in G$, the subset D_t should be an ideal and the map α_t should be a ring (algebra) isomorphism. In the topological setting, each D_t should be an open set and each α_t a homeomorphism, and in the C*-algebra setting each D_t should be a closed ideal and each α_t should be a *-isomorphism.

Associated with a partial action of a group G on a ring A, we have the partial skew group ring, $A \rtimes_\alpha G$, which is the set of all finite formal sums $\sum_{t\in G} a_t\delta_t$, where, for each $t \in G$, $a_t \in D_t$ and δ_t is a symbol. Addition is defined in the usual way and multiplication is determined by $(a_t\delta_t)(b_s\delta_s) = \alpha_t(\alpha_{t^{-1}}(a_t)b_s)\delta_{ts}$. An ideal I of A is said to be *G-invariant* if $\alpha_g(I \cap D_{g^{-1}}) \subseteq I$ holds for all $g \in G$. If A and $\{0\}$ are the only G-invariant ideals of A, then A is said to be *G-simple*.

For $a = \sum_{t\in G} a_t\delta_t \in A \rtimes_\alpha G$, the support of a, which we denote by $\mathrm{supp}(a)$, is the finite set $\{t \in G \ : \ a_t \neq 0\}$, and the cardinality of $\mathrm{supp}(a)$ is denoted by $\#\mathrm{supp}(a)$.