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A generalized Koszul theory and its relation to the classical theory

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ABSTRACT

Let $A = \bigoplus_{i \geqslant 0} A_i$ be a graded locally finite k-algebra where A_0 is a finite dimensional algebra whose finitistic dimension is 0. In this paper we develop a generalized Koszul theory preserving many classical results, and show an explicit correspondence between this generalized theory and the classical theory. Applications in representations of certain categories and extension algebras of standard modules of standardly stratified algebras are described.

1. Introduction

The classical Koszul theory plays an important role in the representation theory of graded algebras. However, there are a lot of structures (algebras, categories, etc.) having natural gradings with non-semisimple degree 0 parts, to which the classical theory

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cannot apply. Particular examples of such structures include tensor algebras generated by non-semisimple algebras A_0 and (A_0, A_0)-bimodules A_1, extension algebras of finitely generated modules (among which we are most interested in extension algebras of standard modules of standardly stratified algebras [8,15]), graded modular skew group algebras, category algebras of finite EI categories [14,27,28], and certain graded k-linear categories. Therefore, it is reasonable to develop a generalized Koszul theory to study representations and homological properties of the above structures.

In [11,18,19,29] several generalized Koszul theories have been described, where the degree 0 part A_0 of a graded algebra A is not required to be semisimple. In [29], A is supposed to be both a left projective A_0-module and a right projective A_0-module. However, in many cases A is indeed a left projective A_0-module, but not a right projective A_0-module. In Madsen's paper [19], A_0 is supposed to have finite global dimension. This requirement is too strong for us since in many applications A_0 is a self-injective algebra or a direct sum of local algebras, and hence A_0 has finite global dimension if and only if it is semisimple, falling into the framework of the classical theory. The theory developed by Green, Reiten and Solberg in [11] works in a very general framework, and we want to find some conditions which are easy to check in practice. The author has already developed a generalized Koszul theory in [17] under the assumption that A_0 is self-injective, and used it to study representations and homological properties of certain categories.

The goal of the work described in this paper is to loosen the assumption that A_0 is self-injective (as required in [17]) and replace it by a weaker condition so that the generalized theory can apply to more situations. Specifically, since we are interested in the extension algebras of modules, category algebras of finite EI categories, and graded k-linear categories for which the endomorphism algebra of each object is a finite dimensional local algebra, this weaker condition should be satisfied by self-injective algebras and finite dimensional local algebras. On the other hand, we also expect that many classical results as the Koszul duality can be preserved. Moreover, we hope to get a close relation between this generalized theory and the classical theory.

A trivial observation tells us that in the classical setup A_0 is semisimple if and only if $\operatorname{gl.dim} A_0$, the global dimension of A_0, is 0. Therefore, it is natural to consider the condition that $\operatorname{fin.dim} A_0$, the *finitistic dimension* of A_0, is 0. Obviously, finite dimensional local algebras and self-injective algebras do have this property. It turns out that this weaker condition is suitable for our applications, and many classical results still hold.

Explicitly, let $A = \bigoplus_{i \geqslant 0} A_i$ be a graded *locally finite* k-algebra generated in degrees 0 and 1, i.e., $\dim_k A_i < \infty$ and $A_1 \cdot A_i = A_{i+1}$ for all $i \geqslant 0$. We assume that both $\operatorname{fin.dim} A_0$ and $\operatorname{fin.dim} A_0^{\mathrm{op}}$ are 0, where A_0^{op} is the opposite algebra of A_0. We then define *generalized Koszul modules* and *generalized Koszul algebras* by linear projective resolutions, as people did for classical Koszul modules and classical Koszul algebras.

It is well known that in the classical theory *linear modules* (defined by linear projective resolutions) and *Koszul modules* (defined by a certain extension property) coincide. We have a similar result: