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Completeness in partial differential algebraic geometry

James Freitag

Department of Mathematics, University of California, Berkeley, 970 Evans Hall, Berkeley, CA 94720-3840, United States

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ABSTRACT

This paper is part of the model theory of fields of characteristic 0, equipped with m commuting derivation operators ($DCF_{0,m}$). It continues to partial differential fields work begun by Wai-Yan Pong, who treated the case $m = 1$. We study the concept of completeness in differential algebraic geometry, applying methods of model theory and differential algebra. Our central tool in applying the valuative criterion developed in differential algebra by E.R. Kolchin, Peter Bloom, and Sally Morrison is a fundamental theorem in classical elimination theory due to the model theorist Lou van den Dries. We use this valuative criterion to give a new family of complete differential algebraic varieties. In addition to completeness, we prove some embedding theorems for differential algebraic varieties of arbitrary differential transcendence degree. As a special case, we show that every differential algebraic subvariety of the projective line which has Lascar rank less than ω^m can be embedded in the affine line.

The theory of differential fields of characteristic zero with finitely many commuting derivations $\delta_1, \ldots, \delta_m$ has a model companion, the theory of differentially closed fields, denoted by $DCF_{0,m}$. The theory is characterized by the property: F is differentially

E-mail address: freitag@math.berkeley.edu.

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closed if and only if any system of partial differential equations over f which has a solution in some differential field extension of F already has a solution in F. Seeing that this property is first order is, of course, nontrivial. In this paper, we fix a large saturated enough model $\mathcal{U} \models DCF_{0,m}$. In the notation of Kolchin, $\mathcal{U}$ is universal over F or any of the differential fields which we consider in this paper; for instance, one could assume without loss of generality that every differential subfield we consider in this paper is of cardinality less than κ and that $\mathcal{U}$ is κ-saturated [15, for a reference on the basics of saturation].

The model theory of partial differential fields was developed in [17], where the basic properties of this theory were originally investigated. Throughout this paper, all differential fields will be assumed to be subfields of $\mathcal{U}$; this is only a matter of convenience and does not affect any of the results. Throughout the paper, we will let F denote a fixed arbitrary differentially closed subfield F of $\mathcal{U}$. We mention briefly that because this theory $DCF_{0,m}$ has quantifier elimination, the F-definable sets in $\mathcal{U}^n$ correspond to the F-constructible sets in the *Kolchin topology*, a differential version of the Zariski topology where the closed sets are given by the vanishing of differential polynomials over F. This fixed Δ-field F will serve throughout the paper, and unless specifically noted, Δ-algebraic varieties are assumed to be defined over F, Δ-rings are Δ–F-algebras, and ring homomorphisms are morphisms of Δ–F-algebras.

The differential varieties we consider will be given by closed subsets in affine or projective space; since we have fixed a large saturated model $\mathcal{U}$ throughout the paper, we are (wlog) considering points with coordinates in $\mathcal{U}$. So, $\mathbb{P}^n$ means $\mathbb{P}^n(\mathcal{U})$ (similarly for affine space). When we are considering points with coordinates in some other field or ring R, we will use the notation $\mathbb{P}^n(R)$. Generalizations to differential schemes are of interest, but are not treated here, see [13,14,9, for instance]. Pillay also considers differential completeness for a slightly different category in [24]; the precise relationship between the notions is not clear, but completeness in the category of D-varieties seems more akin to the notion in the algebraic category. Generalizing this work to the difference-differential topology is of interest, but there are important model-theoretic obstacles. Specifically, differential-difference fields do not have quantifier elimination [18]. Nevertheless, the difference-differential category is of particular interest because we know that the natural analogues of Questions 4.11 and 4.12 are actually known to be distinct.

The definition of Δ-completeness 2.4 is a straightforward generalization of the definition of completeness in the category of abstract algebraic varieties (integral separated schemes of finite type over an algebraically closed field). Section 2 is devoted to giving the basic definitions and particular notation of this paper. After the basic examples, we turn to an example of Kolchin in Section 3, which highlights one of the essential differences between Δ-completeness and completeness for abstract algebraic varieties: $\mathbb{P}^n$ is not Δ-complete. We should note that this paper only considers quasiprojective differential algebraic varieties over a differentially closed field. The completeness question for more abstract differential varieties [12] is of interest. In the category of abstract algebraic varieties, there are complete varieties which are not projective