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A strong geometric hyperbolicity property for directed graphs and monoids

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ABSTRACT

We introduce and study a strong "thin triangle" condition for directed graphs, which generalises the usual notion of hyperbolicity for a metric space. We prove that finitely generated left cancellative monoids whose right Cayley graphs satisfy this condition must be finitely presented with polynomial Dehn functions, and hence word problems in $\mathcal{NP}$. Under the additional assumption of right cancellativity (or in some cases the weaker condition of bounded indegree), they also admit algorithms for more fundamentally semigroup-theoretic decision problems such as Green's relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{J}$, $\mathcal{D}$ and the corresponding pre-orders.

In contrast, we exhibit a right cancellative (but not left cancellative) finitely generated monoid (in fact, an infinite class of them) whose Cayley graph is essentially a tree (hence hyperbolic in our sense and probably any reasonable sense), but which is not even recursively presentable. This seems to be strong evidence that no geometric notion of hyperbolicity will be strong enough to yield much information about finitely generated monoids in absolute generality.

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1. Introduction

Over the past half century, combinatorial group theory has been increasing dominated by ideas from geometry. One of the most successful aspects is the theory of *word hyperbolic groups*, in which a simple, combinatorial notion of negative curvature for a group Cayley graph is used to give tight control on the geometric, combinatorial and even computational structure of the group [20]. The question naturally arises of whether these geometric techniques are particular to groups, or if they apply to a wider class of monoids. There are numerous equivalent characterisations of word hyperbolic groups, which lead to different (non-equivalent) ways in which one could define a word hyperbolic monoid. For example, a finitely presented group is word hyperbolic exactly if it has linear *Dehn function*, and this property can also be studied for monoids [10,29]. A beautiful theorem of Gilman [16] characterises hyperbolic groups as those which admit *context-free multiplication tables*: several authors have studied the class of monoids satisfying this and similar conditions [9,11,14,22]. Another approach is to treat a monoid Cayley graph as an undirected graph, and require that it be a hyperbolic [9,13,15] metric space. In general, the directional information in a monoid or semigroup Cayley graph is of crucial importance in understanding the algebraic structure (and most especially the *ideal* structure) of the monoid, and it is unreasonable to expect much control from any condition on the Cayley graph which disregards the directed structure. For example, it is easily seen that any finitely generated semigroup with a zero element will have undirected Cayley graph of bounded diameter 2. However, this approach does seem to have some merit in classes of semigroups with a very restricted ideal structure, such as *completely simple semigroups* [15].

Here we propose a new way in which hyperbolicity conditions can be extended to cancellative monoids, in a way which is geometric but does not artificially impose a metric structure on the monoid. Specifically, we introduce (in Section 2 below) and study a strong "directed thin triangle" condition for *directed* graphs, which generalises the usual notion of hyperbolicity for a metric space. We prove (in Section 4) that a finitely generated left cancellative monoid whose Cayley graph satisfies this condition must be finitely presented with polynomial Dehn function, and hence solvable (indeed, non-deterministic polynomial time) word problem. Under additional right cancellativity assumptions, we also show (Section 6) that such a monoid admits algorithms for more fundamentally semigroup-theoretic decision problems, including Green's equivalence relations $\mathcal{L}$, $\mathcal{R}$, $\mathcal{J}$, and $\mathcal{D}$ and corresponding pre-orders. These results suggest that, for at least the class of left cancellative monoids, our thin triangle condition carries with it some genuinely "hyperbolic" structure. Truly geometric methods for left cancellative monoids (see for example [19]) are of considerable interest, for example because the word problem for