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Artinian Gorenstein algebras with linear resolutions

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A B S T R A C T

For each pair of positive integers n, d, we construct a complex $\widetilde{\mathbb{G}}'(n)$ of modules over the bi-graded polynomial ring $\widetilde{R} = \mathbb{Z}[x_1, \dots, x_d, \{t_M\}]$, where M roams over all monomials of degree $2n-2$ in $\{x_1, \dots, x_d\}$. The complex $\widetilde{\mathbb{G}}'(n)$ has the following universal property. Let P be the polynomial ring $\boldsymbol{k}[x_1, \dots, x_d]$, where $\boldsymbol{k}$ is a field, and let $\mathbb{I}_n^{[d]}(\boldsymbol{k})$ be the set of homogeneous ideals I in P, which are generated by forms of degree n, and for which P/I is an Artinian Gorenstein algebra with a linear resolution. If I is an ideal from $\mathbb{I}_n^{[d]}(\boldsymbol{k})$, then there exists a homomorphism $\widetilde{R} \to P$, so that $P \otimes_{\widetilde{R}} \widetilde{\mathbb{G}}'(n)$ is a minimal homogeneous resolution of P/I by free P-modules. The construction of $\widetilde{\mathbb{G}}'(n)$ is equivariant and explicit. We give the differentials of $\widetilde{\mathbb{G}}'(n)$ as well as the modules. On the other hand, the homology of $\widetilde{\mathbb{G}}'(n)$ is unknown as are the properties of the modules that comprise $\widetilde{\mathbb{G}}'(n)$. Nonetheless, there is an ideal $\widetilde{I}$ of $\widetilde{R}$ and an element $\boldsymbol{\delta}$ of $\widetilde{R}$ so that $\widetilde{I}\widetilde{R}_{\boldsymbol{\delta}}$ is a Gorenstein ideal of $\widetilde{R}_{\boldsymbol{\delta}}$ and $\widetilde{\mathbb{G}}'(n)_{\boldsymbol{\delta}}$ is a resolution of $\widetilde{R}_{\boldsymbol{\delta}}/\widetilde{I}\widetilde{R}_{\boldsymbol{\delta}}$ by projective $\widetilde{R}_{\boldsymbol{\delta}}$-modules.

The complex $\widetilde{\mathbb{G}}'(n)$ is obtained from a less complicated complex $\widetilde{\mathbb{G}}(n)$ which is built directly, and in a polynomial manner, from the coefficients of a generic Macaulay inverse system Φ. Furthermore, $\widetilde{I}$ is the ideal of $\widetilde{R}$ determined

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Resolutions

by Φ. The modules of $\widetilde{\mathbb{G}}(n)$ are Schur and Weyl modules corresponding to hooks. The complex $\widetilde{\mathbb{G}}(n)$ is bi-homogeneous and every entry of every matrix in $\widetilde{\mathbb{G}}(n)$ is a monomial.

If $m_1, \dots, m_N$ is a list of the monomials in $x_1, \dots, x_d$ of degree $n-1$, then $\boldsymbol{\delta}$ is the determinant of the $N \times N$ matrix $(t_{m_i m_j})$.

The previously listed results exhibit a flat family of $\boldsymbol{k}$-algebras parameterized by $\mathbb{I}_n^{[d]}(\boldsymbol{k})$:

$$\boldsymbol{k}\big[\{t_M\}\big]_{\boldsymbol{\delta}} \to \left(\frac{\boldsymbol{k} \otimes_{\mathbb{Z}} \widetilde{R}}{\widetilde{I}}\right)_{\boldsymbol{\delta}}. \qquad (*)$$

Every algebra P/I, with $I \in \mathbb{I}_n^{[d]}(\boldsymbol{k})$, is a fiber of $(*)$. We simultaneously resolve all of these algebras P/I.

The natural action of $\mathrm{GL}_d(\boldsymbol{k})$ on P induces an action of $\mathrm{GL}_d(\boldsymbol{k})$ on $\mathbb{I}_n^{[d]}(\boldsymbol{k})$. We prove that if $d = 3$, $n \geq 3$, and the characteristic of $\boldsymbol{k}$ is zero, then $\mathbb{I}_n^{[d]}(\boldsymbol{k})$ decomposes into at least four disjoint, non-empty orbits under this group action.

Contents

0. Introduction

Fix a pair of positive integers d and n. We create a ring $\widetilde{R}$ and a complex $\widetilde{\mathbb{G}}'(n)$ of $\widetilde{R}$-modules with the following universal property. Let $P = \boldsymbol{k}[x_1, \dots, x_d]$ be a polynomial ring in d variables over the field $\boldsymbol{k}$ and let I be a grade d Gorenstein ideal in P which is generated by homogeneous forms of degree n. If the resolution of P/I by free P-modules is (Gorenstein) linear, then there exists a ring homomorphism $\widehat{\phi} : \widetilde{R} \to P$ such that $P \otimes_{\widetilde{R}} \widetilde{\mathbb{G}}'(n)$ is a minimal homogeneous resolution of P/I by free P-modules. Our construction is coordinate free.

We briefly describe our construction, many more details will be given later. Let U be a free Abelian group of rank d. The ring $\widetilde{R}$ is equal to

$$\mathrm{Sym}_\bullet^{\mathbb{Z}}\big(U \oplus \mathrm{Sym}_{2n-2}^{\mathbb{Z}} U\big),$$