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Attached primes of local cohomology modules under localization and completion ☆

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A B S T R A C T

Let $(R,\mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module. Following I.G. Macdonald [8], the set of all attached primes of the Artinian local cohomology module $H^i_{\mathfrak{m}}(M)$ is denoted by $\mathrm{Att}_R(H^i_{\mathfrak{m}}(M))$. In [13, Theorem 3.7], R.Y. Sharp proved that if R is a quotient of a Gorenstein local ring then the shifted localization principle holds true for any local cohomology modules $H^i_{\mathfrak{m}}(M)$, i.e.

$$\begin{aligned}&\mathrm{Att}_{R_{\mathfrak{p}}}\big(H^{i-\dim(R/\mathfrak{p})}_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}})\big)\\ &\quad=\big\{\mathfrak{q}R_{\mathfrak{p}}\ \big|\ \mathfrak{q}\in\mathrm{Att}_R H^i_{\mathfrak{m}}(M),\ \mathfrak{q}\subseteq\mathfrak{p}\big\}\end{aligned}\tag{1}$$

for any $\mathfrak{p}\in\mathrm{Spec}(R)$. In this paper, we improve Sharp's result as follows: the shifted localization principle holds true if and only if R is universally catenary and all its formal fibers are Cohen–Macaulay, if and only if the shifted completion principle

$$\mathrm{Att}_{\widehat{R}}\big(H^i_{\mathfrak{m}}(M)\big)=\bigcup_{\mathfrak{p}\in\mathrm{Att}_R(H^i_{\mathfrak{m}}(M))}\mathrm{Ass}_{\widehat{R}}(\widehat{R}/\mathfrak{p}\widehat{R})\tag{2}$$

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holds true for any local cohomology module $H^i_{\mathfrak{m}}(M)$. This also improves the main result of the paper by T.D.M. Chau and the first author [2].

1. Introduction

Throughout this paper, let $(R, \mathfrak{m})$ be a Noetherian local ring and M a finitely generated R-module with $\dim M = d$. It is well known that

$$\operatorname{Ass}_{R_\mathfrak{p}}(M_\mathfrak{p}) = \{\mathfrak{q}R_\mathfrak{p} \mid \mathfrak{q} \in \operatorname{Ass}_R M,\ \mathfrak{q} \subseteq \mathfrak{p}\}$$

for every prime ideal $\mathfrak{p}$ of R. For an Artinian R-module A, the set of all attached primes $\operatorname{Att}_R A$ defined by I.G. Macdonald [8] plays an important role, which is similar to the role of the set of associated primes $\operatorname{Ass}_R M$ of a finitely generated R-module M. It is well known that the local cohomology module $H^i_{\mathfrak{m}}(M)$ is Artinian for all $i \geq 0$. Therefore, it is natural to ask whether the analogous relation

$$\operatorname{Att}_{R_\mathfrak{p}}\big(H^{i-\dim(R/\mathfrak{p})}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p})\big) = \big\{\mathfrak{q}R_\mathfrak{p} \bigm| \mathfrak{q} \in \operatorname{Att}_R\big(H^i_{\mathfrak{m}}(M)\big),\ \mathfrak{q} \subseteq \mathfrak{p}\big\} \tag{1}$$

between $\operatorname{Att}_{R_\mathfrak{p}}(H^{i-\dim(R/\mathfrak{p})}_{\mathfrak{p}R_\mathfrak{p}}(M_\mathfrak{p}))$ and $\operatorname{Att}_R(H^i_{\mathfrak{m}}(M))$ holds true for every $H^i_{\mathfrak{m}}(M)$ and every $\mathfrak{p} \in \operatorname{Spec}(R)$. If R is a quotient of a Gorenstein local ring, R.Y. Sharp [13, Theorem 3.7] proved that the *shifted localization principle* (1) holds true (see also [1, 11.3.2]). However, it is not the case in general, cf. [1, Example 11.3.14].

Another question is about the relation between the attached primes of $H^i_{\mathfrak{m}}(M)$ over R and that of $H^i_{\mathfrak{m}}(M)$ over the $\mathfrak{m}$-adic completion $\widehat{R}$ of R. Denote by $\widehat{M}$ the $\mathfrak{m}$-adic completion of M. Then we have the following well-known relations between $\operatorname{Ass}_R M$ and $\operatorname{Ass}_{\widehat{R}} \widehat{M}$

$$\operatorname{Ass}_R M = \{\mathfrak{P} \cap R \mid \mathfrak{P} \in \operatorname{Ass}_{\widehat{R}} \widehat{M}\} \quad \text{and} \quad \operatorname{Ass}_{\widehat{R}} \widehat{M} = \bigcup_{\mathfrak{p} \in \operatorname{Ass}_R M} \operatorname{Ass}_{\widehat{R}}(\widehat{R}/\mathfrak{p}\widehat{R}),$$

cf. [9, Theorem 23.2]. For an Artinian R-module A, we note that A has a natural structure as an Artinian $\widehat{R}$-module. Moreover, $\operatorname{Att}_R A = \{\mathfrak{P} \cap R \mid \mathfrak{P} \in \operatorname{Att}_{\widehat{R}} A\}$ (see [1, 8.2.4, 8.2.5]), which is in some sense dual to the above first relation between $\operatorname{Ass}_R M$ and $\operatorname{Ass}_{\widehat{R}} \widehat{M}$. However, the second analogous relation may not hold true even when $A = H^i_{\mathfrak{m}}(M)$, i.e. in general the *shifted completion principle* for the local cohomology module $H^i_{\mathfrak{m}}(M)$

$$\operatorname{Att}_{\widehat{R}}\big(H^i_{\mathfrak{m}}(M)\big) = \bigcup_{\mathfrak{p} \in \operatorname{Att}_R(H^i_{\mathfrak{m}}(M))} \operatorname{Ass}_{\widehat{R}}(\widehat{R}/\mathfrak{p}\widehat{R}) \tag{2}$$

is not true, cf. [2, Example 2.3]. Note that if R is a quotient of a Gorenstein local ring, then (2) holds true for any local cohomology module $H^i_{\mathfrak{m}}(M)$, cf. [2, Proposition 2.6].