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Poisson cohomology of two Fano threefolds

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A R T I C L E I N F O

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A B S T R A C T

We study the variety of Poisson structures and compute Poisson cohomology for two families of Fano threefolds – smooth cubic threefolds and the del Pezzo quintic threefold. Along the way we reobtain by a different method earlier results of Loray, Pereira and Touzet in the special case we are considering.

0. Introduction

A *Poisson structure* on a smooth algebraic variety X is given by a bivector field $\omega \in H^0(X, \wedge^2 T_X)$, which satisfies the condition

$$[\omega, \omega] = 0, \tag{$\star$}$$

where [,] denotes the Schouten bracket.

It is natural to consider the variety of Poisson structures $\mathcal{P}$ on X, which is the subvariety in $\mathbb{P}(H^0(X, \wedge^2 T_X))$ given by the equations ($\star$).

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Given any point $\omega \in \mathcal{P}$, one defines Poisson cohomology $H^\star_{Poisson}(X, \omega)$ of X with respect to ω [9].

Explicit computation of Poisson cohomology for various algebraic varieties is of interest (see [3]). In [3] Wei Hong and Ping Xu computed Poisson cohomology of del Pezzo surfaces. The condition $(\star)$ is automatically satisfied in that case for dimension reasons. This is not the case for Fano threefolds.

Poisson Fano threefolds with Picard number 1 were classified in [10,11]. They are exactly the following (see [10,11]):

- the projective space $\mathbb{P}^3$,
- the quadric $Q \subset \mathbb{P}^4$,
- a sextic hypersurface $X \subset \mathbb{P}(1, 1, 1, 2, 3)$,
- a quartic hypersurface $X \subset \mathbb{P}(1, 1, 1, 1, 2)$,
- a cubic hypersurface $X \subset \mathbb{P}^4$,
- a complete intersection of two quadrics in $\mathbb{P}^5$,
- the del Pezzo quintic threefold $X \subset \mathbb{P}^6$,
- the Mukai–Umemura threefold.

In each case Loray, Pereira and Touzet described the dimensions of the irreducible components of the variety of Poisson structures, described their smoothness and some other properties (see [10,11]).

In this note we consider (from a different viewpoint) two families of Poisson Fano threefolds from the above list: cubic threefolds and the del Pezzo quintic threefold. For each of them we describe explicitly the variety of Poisson structures (thus reobtaining the results of [10,11] in a special case) and compute Poisson cohomology.

Our main results are the following. For the explicit computations of matrices A_ω, B_ω and C_ω we refer the reader to Sections 2.3 and 3.2.

Theorem 1. *(See Loray, Pereira, Touzet, [10,11].) Let X be a smooth cubic threefold. Then the variety of Poisson structures $\mathcal{P} \subset \mathbb{P}(H^0(X, \wedge^2 T_X))$ on X is the Grassmannian $G(2, 5)$ of lines in $\mathbb{P}^4$ and $\mathcal{P} \subset \mathbb{P}(H^0(X, \wedge^2 T_X))$ is the Plücker embedding.*

Theorem 2. *Let X be a smooth cubic threefold and $\omega \in \mathcal{P} \subset \mathbb{P}(H^0(X, \wedge^2 T_X))$ a Poisson structure on X. Then the dimensions of the Poisson cohomology groups are as follows:*

- $dim(H^0_{Poisson}(X, \omega)) = 1$,
- $dim(H^1_{Poisson}(X, \omega)) = 0$,
- $dim(H^2_{Poisson}(X, \omega)) = 20 - rk(C_\omega)$,
- $dim(H^3_{Poisson}(X, \omega)) = 15 - rk(C_\omega)$.

Theorem 3. *(See Loray, Pereira, Touzet, [10,11].) Let X be the (smooth) del Pezzo quintic threefold. Then the variety of Poisson structures $\mathcal{P} \subset \mathbb{P}(H^0(X, \wedge^2 T_X))$ on X is the dis-*