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Derivations of a parametric family of subalgebras of the Weyl algebra

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ABSTRACT

An Ore extension over a polynomial algebra $\mathbb{F}[x]$ is either a quantum plane, a quantum Weyl algebra, or an infinite-dimensional unital associative algebra A_h generated by elements x, y, which satisfy $yx-xy = h$, where $h \in \mathbb{F}[x]$. When $h \neq 0$, the algebra A_h is subalgebra of the Weyl algebra A_1 and can be viewed as differential operators with polynomial coefficients. This paper determines the derivations of A_h and the Lie structure of the first Hochschild cohomology group $\mathsf{HH}^1(\mathsf{A}_h) = \mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)/\mathsf{Inder}_{\mathbb{F}}(\mathsf{A}_h)$ of outer derivations over an arbitrary field. In characteristic 0, we show that $\mathsf{HH}^1(\mathsf{A}_h)$ has a unique maximal nilpotent ideal modulo which $\mathsf{HH}^1(\mathsf{A}_h)$ is 0 or a direct sum of simple Lie algebras that are field extensions of the one-variable Witt algebra. In positive characteristic, we obtain decomposition theorems for $\mathsf{Der}_{\mathbb{F}}(\mathsf{A}_h)$ and $\mathsf{HH}^1(\mathsf{A}_h)$ and describe the structure of $\mathsf{HH}^1(\mathsf{A}_h)$ as a module over the center of A_h.

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1. Introduction

We consider a family of infinite-dimensional unital associative algebras A_h parametrized by a polynomial h in one variable, with the definition given as follows:

Definition 1.1. Let $\mathbb{F}$ be a field, and let $h \in \mathbb{F}[x]$. The algebra A_h is the unital associative algebra over $\mathbb{F}$ with generators x, y and defining relation $yx = xy + h$ (equivalently, $[y, x] = h$ where $[y, x] = yx - xy$).

These algebras arose naturally in considering Ore extensions over a polynomial algebra $\mathbb{F}[x]$. Many algebras can be realized as iterated Ore extensions, and for that reason, Ore extensions have become a mainstay in associative theory. Recall that an Ore extension $\mathsf{A} = \mathsf{R}[y, \sigma, \delta]$ is built from a unital associative (not necessarily commutative) algebra R over a field $\mathbb{F}$, an $\mathbb{F}$-algebra endomorphism σ of R, and a σ-derivation of R, where by a σ-derivation δ we mean that δ is $\mathbb{F}$-linear and $\delta(rs) = \delta(r)s+\sigma(r)\delta(s)$ holds for all $r, s \in \mathsf{R}$. Then $\mathsf{A} = \mathsf{R}[y, \sigma, \delta]$ is the algebra generated by y over R subject to the relation

$$yr = \sigma(r)y + \delta(r) \quad \text{for all } r \in \mathsf{R}.$$

Under the assumption that $\mathsf{R} = \mathbb{F}[x]$ and σ is an automorphism of R, the following result holds. (Compare [3] and [1], which have a somewhat different division into cases.)

Lemma 1.2. *Assume $\mathsf{A} = \mathsf{R}[y, \sigma, \delta]$ is an Ore extension with $\mathsf{R} = \mathbb{F}[x]$, a polynomial algebra over a field $\mathbb{F}$ of arbitrary characteristic, and σ an automorphism of R. Then A is isomorphic to one of the following:*

(a) *a quantum plane;*
(b) *a quantum Weyl algebra;*
(c) *an algebra A_h with generators x, y and defining relation $yx = xy + h$ for some polynomial $h \in \mathbb{F}[x]$.*

The algebras A_h result from taking $\mathsf{R} = \mathbb{F}[x]$, σ to be the identity automorphism, and $\delta : \mathsf{R} \to \mathsf{R}$ to be the derivation given by

$$\delta(f) = f'h, \tag{1.3}$$

where f' is the usual derivative of f with respect to x.

Quantum planes and quantum Weyl algebras are examples of generalized Weyl algebras in the sense of [4, 1.1], and as such, have been studied extensively. In [5,6], we determined the center, normal elements, and prime ideals of the algebras A_h, as well as the automorphisms and their invariants, isomorphisms between two algebras A_g and A_h, and the irreducible A_h-modules over any field $\mathbb{F}$. Our aim in this paper is to compute the derivations and first Hochschild cohomology group of the algebras A_h over an arbitrary field.