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Associating vertex algebras with the unitary Lie algebra

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A R T I C L E I N F O

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A B S T R A C T

In this paper, we associate vertex algebras and their two different kinds of module categories with the unitary Lie algebra $\widehat{\mathfrak{u}}_N(\mathbb{C}_{\widetilde{\Gamma}})$ for $N \geq 2$ being a positive integer and $\widetilde{\Gamma} = \{q^n \mid n \in \mathbb{Z}\}$, where the nonzero complex number q is not a root of unity. It is proved that for any complex number ℓ, the category of restricted $\widehat{\mathfrak{u}}_N(\mathbb{C}_{\widetilde{\Gamma}})$-modules of level ℓ is canonically isomorphic to the category of quasi modules for certain vertex algebra. And we also prove that the category of restricted $\widehat{\mathfrak{u}}_N(\mathbb{C}_{\widetilde{\Gamma}})$-modules of level ℓ is isomorphic to the category of Γ-equivariant ϕ-coordinated quasi modules for the same vertex algebra, where Γ is an automorphism group of this vertex algebra.

1. Introduction

Unitary Lie algebras (or Steinberg unitary Lie algebras) were first introduced in [1], which are generalization of the usual loop algebras by replacing the commutative coordinated ring with a noncommutative algebra. In [3], a representation of non-trivial central

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extension $\widehat{\mathfrak{u}}_\nu(\mathcal{R}_\sigma, -)$ of the unitary Lie algebra $\mathfrak{u}_\nu(\mathcal{R}_\sigma, -)$ coordinated by skew Laurent polynomial rings was constructed by vertex operators, where $N \geq 2$ is a positive integer, $\mathcal{R}_\sigma$ is the skew Laurent polynomial ring with an anti-involution $^-$, and σ is a linear character from an abelian group G to the multiplicative group $\mathbb{C}^\times$ (the set of nonzero complex numbers). The unitary Lie algebra we study in this paper is $\widehat{\mathfrak{u}}_N(\mathcal{R}_\sigma, -)$ with $G = \widetilde{\Gamma} = \{q^n \mid n \in \mathbb{Z}\}$ and $\sigma = id$, where the complex number q is not a root of unity. We denote this unitary Lie algebra by $\widehat{\mathfrak{u}}_N(\mathbb{C}_{\widetilde{\Gamma}})$, it is spanned by the elements $e_{ij}(m,n)$, $\mathbf{c}$ for $1 \leq i,j \leq N$, $m,n \in \mathbb{Z}$ with relation $e_{ij}(m,n) = -(-q^m)^{-n} e_{ji}(-m,n)$, and subject to

$$\begin{aligned}
&\big[e_{ij}(m,n), e_{kl}(s,r)\big] \\
&\quad = \delta_{jk}\big(q^m\big)^r e_{il}(m+s,n+r) + \delta_{il}\big(-q^m\big)^{-n-r}\big(q^s\big)^{-r} e_{jk}(-m-s,n+r) \\
&\qquad - \delta_{ik}(-1)^n\big(q^m\big)^{-n-r} e_{jl}(-m+s,n+r) - \delta_{jl}\big(q^m\big)^r\big(-q^s\big)^{-r} e_{ik}(m-s,n+r) \\
&\qquad + \delta_{jk}\delta_{il}\delta_{m+s,0}\delta_{n+r,0} n\mathbf{c} - \delta_{ik}\delta_{jl}\delta_{m-s,0}\delta_{n+r,0}(-1)^n n\mathbf{c}. \qquad (1.1)
\end{aligned}$$

In this paper, we study the unitary Lie algebra in the context of vertex algebras and their corresponding quasi modules. As our main results, we associate the unitary Lie algebras with vertex algebras and two types of quasi modules, one being quasi modules for the vertex algebra viewed as a Γ-vertex algebra (see [9] and [10]), the other being Γ-equivariant ϕ-coordinated quasi modules (see [13]) for the same vertex algebra, where Γ is an automorphism group of the vertex algebra.

It is well known (see [4]; cf. [8]) that certain infinite-dimensional Lie algebras such as affine (Kac–Moody) Lie algebras, Virasoro algebra, and Heisenberg algebras of a certain type can also be canonically associated with vertex algebras and their modules. Recently, some other Lie algebras such as the toroidal Lie algebras, quantum torus Lie algebras, certain deformed Heisenberg algebras and the Lie algebra $\mathfrak{gl}_\infty$ could be associated with vertex algebras or their likes (see [2,15,9,11,7]).

Note that in the association of vertex algebra structures with Lie algebras like quantum torus Lie algebras, a notion of quasi local (especially Γ-local) subset of $\mathcal{E}(W) = \mathrm{Hom}(W, W((x)))$ was introduced in [9], where W is any vector space. Quasi local means that for any two generating functions $a(x)$ and $b(x)$, there exists a nonzero polynomial $f(x_1,x_2)$ such that

$$f(x_1,x_2)a(x_1)b(x_2) = f(x_1,x_2)b(x_2)a(x_1),$$

and Γ-local requires that $f(x_1,x_2)$ should be of the form $(x_1 - \alpha_1 x_2)\cdots(x_1 - \alpha_r x_2)$, for $\alpha_1,\dots,\alpha_r \in \varphi(\Gamma)$, where $\varphi : \Gamma \to \mathbb{C}^\times$ is a group homomorphism. And it was proved that any quasi local (respectively, Γ-local) subset generates a vertex algebra (respectively, Γ-vertex algebra) in a certain canonical way. However, the vector space W under the obvious action is not a module for the vertex algebra, but a what was called quasi module