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Weightless cohomology of algebraic varieties

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Using Morel's weight truncations in categories of mixed sheaves, we attach to varieties over finite fields or the complex numbers a series of groups called the weightless cohomology groups. These lie between the usual cohomology and intersection cohomology, have natural ring structures, and are functorial for certain morphisms.

1. Introduction

Let k be the field $\mathbb{C}$ of complex numbers or a finite field. If k is finite let l be a prime different from the characteristic of k. For a variety X/k we let $H^*(X)$ denote the singular cohomology $H^*(X,\mathbb{Q})$ if $k=\mathbb{C}$ or the étale cohomology $H^*(X_{\bar{k}},\mathbb{Q}_l)$ if k is finite. Similarly, $IH^*(X)$, $H_*^{BM}(X)$, etc. denote the appropriate intersection cohomology,

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Borel–Moore homology, etc. Let $D^b(X)$ denote the derived category of mixed Hodge modules [21] in the case $k = \mathbb{C}$ or the derived category of mixed l-adic complexes [4] if k is finite. $(\cdot)$ denotes the Tate twist.

1.1. Properties of weightless cohomology

Using truncation functors introduced by S. Morel [14], we attach to a variety X/k a complex

$$EC_X \in D^b(X)$$

which we call the *weightless complex* of X. Its hypercohomology groups

$$EH^*(X) := \mathbb{H}^*(X, EC_X)$$

are the *weightless cohomology* groups of X. These are finite-dimensional $\mathbb{Q}$-vector spaces (if $k = \mathbb{C}$) or $\mathbb{Q}_l$-vector spaces (if k is finite) with the following properties (Theorem 4.2.1):

(i) If X is of dimension d then $EH^i(X) \neq 0$ only if $i \in [0, 2d]$. If $\hat{X} \to X$ is the normalization then $EH^*(\hat{X}) = EH^*(X)$. If X is smooth then $EH^*(X) = H^*(X)$.
(ii) $EH^*(X)$ is a graded-commutative $\mathbb{Q}$-algebra (if $k = \mathbb{C}$) or $\mathbb{Q}_l$-algebra (if k is finite).
(iii) $EH^*(X \times Y) \cong EH^*(X) \otimes EH^*(Y)$.
(iv) $EH^*(X)$ lives between the usual cohomology $H^*(X)$, and the intersection cohomology $IH^*(X)$, that is, there are homomorphisms

$$H^*(X) \to EH^*(X) \to IH^*(X)$$

factoring the natural homomorphism $H^*(X) \to IH^*(X)$. The first morphism is a ring homomorphism and the second is an $H^*(X)$-module homomorphism. In general, these homomorphisms are neither injective nor surjective.
(v) If $f : Y \to X$ is a morphism of varieties such that the image of each irreducible component of f meets the smooth locus of X then there is a natural ring homomorphism $f^* : EH^*(X) \to EH^*(Y)$ compatible with $H^*(X) \to H^*(Y)$. If $f : Y \to X$ and $g : Z \to Y$ are morphisms such that f, g, and $f \circ g$ satisfy this condition (that the image of each irreducible component of the domain meets the smooth locus of the codomain) then $(f \circ g)^* = g^* \circ f^*$.
(vi) If $d = \dim X$ then $EH^{2d}(X) = IH^{2d}(X)$ and $EH^{2d-1}(X) \twoheadrightarrow IH^{2d-1}(X)$.
(vii) There is a module action of $EH^*(X)$ on $IH^*(X)$ factoring the action of $H^*(X)$ on $IH^*(X)$.

These statements reflect properties of the complex EC_X in $D^b(X)$.

If $k = \mathbb{C}$ then $EH^*(X)$ carries a (rational) mixed Hodge structure. If k is finite then $EH^*(X)$ has an action of $\mathrm{Gal}(\bar{k}/k)$ and hence a weight filtration coming from the