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Blocks with transitive fusion systems ☆

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Suppose that all nontrivial subsections of a p-block B are conjugate (where p is a prime). By using the classification of the finite simple groups, we prove that the defect groups of B are either extraspecial of order p^3 with $p \in \{3, 5\}$ or elementary abelian.

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1. Introduction

Let p be a prime, and let $\mathcal{F}$ be a saturated fusion system on a finite p-group P (cf. [1] and [5]). We call $\mathcal{F}$ *transitive* if any two nontrivial elements in P are $\mathcal{F}$-conjugate. In this case, P has exponent $\exp(P) \le p$, and $\mathrm{Aut}_{\mathcal{F}}(P)$ acts transitively on $\mathrm{Z}(P) \setminus \{1\}$. This paper is motivated by the following:

Conjecture 1.1. *(Cf. [19].) Let $\mathcal{F}$ be a transitive fusion system on a finite p-group P where p is a prime. Then P is either extraspecial of order p^3 or elementary abelian.*

Moreover, if P is extraspecial of order p^3 then results by Ruiz and Viruel [22] imply that $p \in \{3, 5, 7\}$. Note that the conjecture is trivially true for $p = 2$ since groups of exponent 2 are abelian. Thus Conjecture 1.1 is only of interest for $p > 2$. The aim of this paper is to prove the conjecture above for saturated fusion systems coming from blocks.

Theorem 1.2. *Let p be a prime, and let B be a p-block of a finite group G with defect group P. If the fusion system $\mathcal{F} = \mathcal{F}_P(B)$ of B on P is transitive then P is either extraspecial of order p^3 or elementary abelian.*

If P is extraspecial of order p^3 then the results in [22] and [16] imply that $p \in \{3, 5\}$. We call a block B with defect group P and transitive fusion system $\mathcal{F}_P(B)$ *fusion-transitive*. Whenever B has full defect then the theorem is a consequence of the results in [19]. In our proof of the theorem above, we will make use of the classification of the finite simple groups.

2. Saturated fusion systems

We begin with some results on arbitrary saturated fusion systems.

Proposition 2.1. *Let p be a prime, and let $\mathcal{F}$ be a transitive fusion system on a finite p-group P where $|P| \ge p^4$. Suppose that P contains an abelian subgroup of index p. Then P is abelian.*

Proof. We assume the contrary. Then $p > 2$.

Suppose first that P contains two distinct abelian subgroups A, B of index p. Then $AB = P$, $A \cap B \subseteq \mathrm{Z}(P)$ and $|P : A \cap B| = p^2$. Since P is nonabelian we conclude that $|P : \mathrm{Z}(P)| = p^2$. Thus $1 \ne P' \subseteq \mathrm{Z}(P)$. Since $\mathrm{Aut}_{\mathcal{F}}(P)$ acts transitively on $\mathrm{Z}(P) \setminus \{1\}$, we conclude that $P' = \mathrm{Z}(P)$. Hence there are $x, y \in P$ such that $P = \langle x, y \rangle$. Then $P' = \langle [x, y] \rangle$ (cf. III.1.11 in [13]); in particular, we have $|P'| = p$ and $|P| = p^3$, a contradiction.

It remains to consider the case where P contains a unique abelian subgroup A of index p. Let Z be a subgroup of order p in $\mathrm{Z}(P)$, and let B be an arbitrary subgroup of order p in A. By transitivity, there is an isomorphism $\phi : B \to Z$ in $\mathcal{F}$. By definition,