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The maximum cardinality of minimal inversion complete sets in finite reflection groups

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We compute for reflection groups of type A, B, D, F_4, H_3 and for dihedral groups a statistic counting the maximal cardinality of a set of elements in the group whose generalized inversions yield the full set of inversions and which are minimal with respect to this property. We also provide lower bounds for the E types that we conjecture to be the exact value of our statistic.

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1. Introduction

Let S_n be the symmetric group on $\{1, \ldots, n\}$. Recall that if $\sigma \in S_n$, $1 \leq i < j \leq n$ and $\sigma(i) > \sigma(j)$, the pair $(\sigma(i), \sigma(j))$ is said to be an inversion for $\sigma \in S_n$. Let $N(\sigma)$ denote the inversion set of σ. Also set $D = \{(i, j) \mid 1 \leq j < i \leq n\}$.

Definition 1.1. We say that $Y \subset S_n$ is *inversion complete* if $\bigcup_{\sigma \in Y} N(\sigma) = D$, and that is *minimal inversion complete* if it is inversion complete and minimal with respect to this property.

The following result in extremal combinatorics of permutations has been communicated to us by Fabio Tardella [13], who informed us about a forthcoming joint work with M. Queyranne and E. Balandraud.

Theorem 1.1. *The maximal cardinality of a minimal inversion complete subset of S_n is $\lfloor \frac{n^2}{4} \rfloor$.*

The enumerative problem dealt with in Theorem 1.1 admits a natural generalization to finite reflection groups. Indeed, let Δ be a finite crystallographic irreducible root system and W be the corresponding Weyl group. Fix a set of positive roots $\Delta^+ \subset \Delta$. If $w \in W$, then permutation inversions are naturally replaced by the subset $N(w)$ of Δ^+ defined in (2.1) (cf. [2]). The problem consists in determining the maximal cardinality of a subset Y of W such that $\bigcup_{w \in Y} N(w) = \Delta^+$ and minimal with respect to this property. Let us denote this number by $MC(T)$ for a group of type T.

In this perspective we provide a proof of Theorem 1.1 and of the following results.

Theorem 1.2. $MC(B_n) = \binom{n}{2} + 1$.

Theorem 1.3. $MC(D_n) = \binom{n}{2}$.

We also have proofs that

$$MC(F_4) \geq 6, \qquad MC(E_6) \geq 16, \qquad MC(E_7) \geq 27, \qquad MC(E_8) \geq 36. \tag{1.1}$$

We conjecture that these bounds are actually the exact values of our statistic. We have indeed a computer assisted proof of equality in type F_4 (cf. Remark 7.1).

By using the canonical root system (cf. Section 8), our problem further generalizes to noncrystallographic types. It is quite easy to prove that $MC(I_2(m)) = 2$; in particular, since G_2 is $I_2(6)$, we have $MC(G_2) = 2$. We can also show that $MC(H_3) = 5$ and that $MC(H_4) \geq 8$. (See Section 8.)

Theorems 1.1, 1.2, 1.3 are proven in two steps. First we exhibit a minimal inversion complete set of the desired cardinality, which gives a lower bound for $MC(T)$. The choice of this set is motivated by the theory of abelian ideals of Borel subalgebras. Then we prove