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$\mathcal{W}_n^+$- and $\mathcal{W}_n$-module structures on $U(\mathfrak{h}_n)$

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ABSTRACT

Let $\mathfrak{h}_n$ be the Cartan subalgebra of the Witt algebras $\mathcal{W}_n^+ = \operatorname{Der}\mathbb{C}[t_1, t_2, ..., t_n]$ and $\mathcal{W}_n = \operatorname{Der}\mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \cdots, t_n^{\pm 1}]$ where $1 \leq n \leq \infty$. In this paper, we classify the modules over $\mathcal{W}_n^+$ and over $\mathcal{W}_n$ which are free $U(\mathfrak{h}_n)$-modules of rank 1. These are the $\mathcal{W}_n^+$-modules $\Omega(\Lambda_n, a, S)$ for some $\Lambda_n = (\lambda_1, \cdots, \lambda_n) \in (\mathbb{C}^*)^n$, $a \in \mathbb{C}$, and $S \subset \{1, 2, ..., n\}$; and $\mathcal{W}_n$-modules $\Omega(\Lambda_n, a)$ for some $\Lambda_n \in (\mathbb{C}^*)^n$ and some $a \in \mathbb{C}$.

1. Introduction

We denote by $\mathbb{Z}$, $\mathbb{Z}_+$, $\mathbb{N}$ and $\mathbb{C}$ the sets of all integers, non-negative integers, positive integers and complex numbers, respectively. All vector spaces and algebras in this paper are over $\mathbb{C}$. We denote by $U(\mathfrak{a})$ the universal enveloping algebra of the Lie algebra $\mathfrak{a}$ over $\mathbb{C}$.

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For any integer n: $1 \leq n \leq \infty$, let $\mathcal{A}_n = \mathbb{C}[t_1^{\pm 1}, t_2^{\pm 1}, \cdots, t_n^{\pm 1}]$ and $\mathcal{A}_n^+ = \mathbb{C}[t_1, t_2, \cdots, t_n]$ be the polynomial algebras. Then we have the Lie algebras $\mathcal{W}_n = \mathrm{Der}(\mathcal{A}_n)$ and $\mathcal{W}_n^+ = \mathrm{Der}(\mathcal{A}_n^+)$. These Lie algebras are known as the Witt algebras of rank n. We know that $\mathcal{W}_1$ is the centerless Virasoro algebra.

The theory of weight Virasoro modules with finite-dimensional weight spaces is fairly well-developed. In 1992, O. Mathieu [15] classified all irreducible modules with finite-dimensional weight spaces over the Virasoro algebra, proving a conjecture of Kac [9]. More precisely, O. Mathieu proved that irreducible weight modules with finite-dimensional weight spaces fall into two classes: (i) highest/lowest weight modules and (ii) modules of tensor fields on a circle and their quotients. V. Mazorchuk and K. Zhao [17] proved that an irreducible weight module over the Virasoro algebra is either a Harish-Chandra module or a module in which all weight spaces in the weight lattice are infinite-dimensional. In [4,5,10,14,26], some simple weight modules over the Virasoro algebra with infinite-weight spaces were constructed. Very recently, the non-weight simple representation theory of the Virasoro algebra has made a big progress. A lot of new non-weight simple modules were obtained in [2,7,12,10,13,16,18,21,23,24] by using different methods.

The theory of weight representations over Witt algebras $\mathcal{W}_n$ for $1 < n < \infty$ has also well-developed. In 2013, Y. Billig and V. Futorny [1] successfully proved the conjecture given by Eswara Rao [6] in 2004. They proved that irreducible modules over $\mathcal{W}_n$ ($1 < n < \infty$) with finite-dimensional weight spaces also fall into two classes: (i) modules of the highest weight type and (ii) modules of tensor fields on a torus and their quotients. V. Mazorchuk and K. Zhao [19] gave an explicit description of the support of an arbitrary irreducible weight module. There are also mixed weight modules over $\mathcal{W}_n$ for $1 < n < \infty$, see, for example [8], while in [11] some irreducible weight modules over $\mathcal{W}_n$ with infinite-dimensional weight spaces were constructed. Recently, in [25] some examples of non-weight irreducible modules over $\mathcal{W}_n$ with $1 < n < \infty$ were given. We have not seen other examples of non-weight irreducible representations over these Witt algebras.

For the Witt algebras $\mathcal{W}_n^+$, I. Penkov and V. Serganova [22] gave an explicit description of the support of an arbitrary irreducible weight module. We have not seen any other studies about irreducible representations over these Witt algebras.

In this paper, we mainly concern about the non-weight representations over Witt algebras $\mathcal{W}_n$ and $\mathcal{W}_n^+$ for $1 \leq n \leq \infty$. More precisely, we will classify a class of non-weight modules which are free $U(\mathfrak{h}_n)$-modules of rank 1, where $\mathfrak{h}_n$ is the Cartan subalgebra of $\mathcal{W}_n$ and $\mathcal{W}_n^+$.

Firstly, let us recall some definitions and some old representations over $\mathcal{W}_n$ and over $\mathcal{W}_n^+$. Then construct some new representations over $\mathcal{W}_n$ and over $\mathcal{W}_n^+$.

Denote by $\mathbb{Z}^\infty$ the set of all sequences $k = (k_1, k_2, k_3, \cdots)$ where $k_i \in \mathbb{Z}$ and only a finite number of k_i can be nonzero. Denote $(\mathbb{C}^*)^\infty = \{(c_k)_{k \in \mathbb{N}} : c_k \in \mathbb{C}^*\}$. Let $\mathbb{Z}_{\geq -1} = \{l \in \mathbb{Z} : l \geq -1\}$. For any $n \in \mathbb{N}$, let

$$\mathbb{Z}^n_{+,i} = \big\{(k_1, \cdots, k_n) \in \mathbb{Z}^n : k_i \in \mathbb{Z}_{\geq -1} \text{ and } k_j \in \mathbb{Z}_+ \text{ for all } j \neq i\big\},$$
$$\mathbb{Z}^\infty_{+,i} = \big\{(k_1, k_2, k_3, \cdots) \in \mathbb{Z}^\infty : k_i \in \mathbb{Z}_{\geq -1} \text{ and } k_j \in \mathbb{Z}_+ \text{ for all } j \neq i\big\}.$$