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Criteria for the existence of a Jordan–Chevalley–Seligman decomposition

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ABSTRACT

Let $(L, [p])$ be a finite dimensional restricted Lie algebra over a field $\mathbb{K}$ of positive characteristic p. A Jordan–Chevalley–Seligman decomposition of $x \in L$ is a unique expression of x as a sum of commuting semisimple and nilpotent elements in L. It is well-known that each $x \in L$ has such a decomposition when $\mathbb{K}$ is perfect. When $\mathbb{K}$ is non-perfect, the present paper gives several criteria for the existence of a Jordan–Chevalley–Seligman decomposition for a given $x \in L$ as well as for determining when an element in the restricted subalgebra generated by x is semisimple or nilpotent.

1. Introduction

Let L be a Lie algebra over a field $\mathbb{K}$ of positive characteristic p with the Lie bracket $[\cdot,\cdot]$. If $x \in L$, then the *adjoint map* $y \mapsto [x, y]$ is denoted by $\operatorname{ad} x$. A pair $(L, [p])$ is a *restricted Lie algebra* if $[p] : L \to L$, $x \mapsto x^{[p]}$ is a unary operation such that, for every $x, y \in L$ and $\alpha \in \mathbb{K}$,

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(R1) $\operatorname{ad} x^{[p]} = (\operatorname{ad} x)^p$;
(R2) $(\alpha x)^{[p]} = \alpha^p x^{[p]}$;
(R3) $(x + y)^{[p]} = x^{[p]} + y^{[p]} + \sum_{k=1}^{p-1} s_k(x, y)$,

where $(\operatorname{ad}(x\otimes X+y\otimes 1))^{p-1}(x\otimes 1) = \sum_{k=1}^{p-1} ks_k(x, y)\otimes X^{k-1}$ in the Lie algebra $L\otimes_{\mathbb{K}}\mathbb{K}[X]$ over the polynomial ring $\mathbb{K}[X]$.

Throughout this paper it is assumed that $\mathbb{K}$ is an *arbitrary* field of positive characteristic p and $(L, [p])$ is a finite dimensional restricted Lie algebra over $\mathbb{K}$.

A *Jordan–Chevalley–Seligman decomposition* (JCS decomposition) of $x \in L$ is a sum $x = s+n$ of $s, n \in L$ such that s is $[p]$-semisimple, n is $[p]$-nilpotent, and $[s, n] = 0$, where the precise notions of $[p]$-semisimplicity and $[p]$-nilpotency are described in the following well-known definition.

Definition 1.1. (See [8, p. 79, p. 67].) Let $x \in L$. One says that:

(i) x is $[p]$-*semisimple* (over $\mathbb{K}$) if $x = \alpha_1 x^{[p]} + \alpha_2 x^{[p]^2} + \cdots + \alpha_r x^{[p]^r}$ for some positive integer r and $\alpha_1, \ldots, \alpha_r \in \mathbb{K}$;
(ii) x is $[p]$-*nilpotent* if $x^{[p]^k} = 0$ for some positive integer k.

In a classical result, Seligman proved (cf. [6, Theorem 1, p. 528], [5, Theorem V.7.2, p. 120]) that each $x \in L$ always has a JCS decomposition when the base field $\mathbb{K}$ of L is *perfect*. Independently, in [7, Theorem 1, p. 26], Schue also proved a slightly sharper form over an *algebraically closed* base field. The common method used in their proofs is to consider the minimal p-equation for a given element in a restricted Lie algebra using so-called *p-polynomials* (*additive polynomials*). Both authors give a sufficient condition under which a JCS decomposition exists, namely, the Frobenius map on $\mathbb{K}$ should be surjective. However, to characterize when an element of a finite dimensional restricted Lie algebra over an *arbitrary* field has a JCS decomposition, one needs more elaborate algebraic tools. In this paper we use several results of Ore [3,4] on the skew polynomial ring structure of the set of all p-polynomials (see the definition of $\mathcal{R}(\mathbb{K})$ in Section 2).

The main goal of the present paper is to generalize Seligman's result from a perfect field to an arbitrary field by giving sufficient and necessary conditions for the existence of a JCS decomposition for any given element in a finite dimensional restricted Lie algebra. In the following we formulate the three different criteria for the existence of a JCS decomposition. We will need the concept of a minimal p-equation. Let us consider a map $M : L \to \mathbb{K}[X]$, $x \mapsto M_x$ such that:

(M1) $M_x = X^{p^n} + \alpha_{n-1} X^{p^{n-1}} + \cdots + \alpha_1 X^p + \alpha_0 X$ for some nonnegative integer n and $\alpha_0, \ldots, \alpha_{n-1} \in \mathbb{K}$;
(M2) $x^{[p]^n} + \alpha_{n-1} x^{[p]^{n-1}} + \cdots + \alpha_1 x^{[p]} + \alpha_0 x = 0$ in L;
(M3) M_x has the minimum degree for which (M1) and (M2) hold.