Journal of Algebra 424 (2015) 390–415

Contents lists available at ScienceDirect

Journal of Algebra

www.elsevier.com/locate/jalgebra

Imaginary Verma modules and Kashiwara algebras for $U_q(\hat{\mathfrak{g}})$ ☆

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ARTICLE INFO

Article history:
Received 8 August 2013
Available online 18 October 2014
Communicated by Masaki Kashiwara

MSC:
17B37
17B67
17B10

Keywords:
Quantum affine algebras
Imaginary Verma modules
Kashiwara algebras
Simple modules

ABSTRACT

We consider imaginary Verma modules for quantum affine algebra $U_q(\hat{\mathfrak{g}})$, where $\hat{\mathfrak{g}}$ has Coxeter–Dynkin diagram of ADE type, and construct Kashiwara type operators and the Kashiwara algebra $\mathcal{K}_q$. We show that a certain quotient $\mathcal{N}_q^-$ of $U_q(\hat{\mathfrak{g}})$ is a simple $\mathcal{K}_q$-module.

☆ The first author would like to thank North Carolina State University and the University of São Paulo for their support and hospitality during his numerous visits to Raleigh and São Paulo. The second author was partially supported by FAPESP (grant 2014/09310-5) and CNPq (grant 301320/2013-6). The third author was partially supported by the NSA grant H98230-12-1-0248 and Simons Foundation Grant 307555.

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http://dx.doi.org/10.1016/j.jalgebra.2014.09.025
0021-8693/

1. Introduction

Let $\hat{\mathfrak{g}}$ be an affine Lie algebra and Δ denote the set of roots with respect to the Cartan subalgebra $\hat{\mathfrak{h}}$. Then we have a natural (standard) partition of $\Delta = \Delta_+ \cup \Delta_-$ into set of positive and negative roots. With respect to this standard partition we have a standard Borel subalgebra from which we may induce the standard Verma module. A partition $\Delta = S \cup -S$ of the root system Δ is said to be a closed partition if, whenever $\alpha, \beta \in S$ and $\alpha + \beta \in \Delta$, we have $\alpha + \beta \in S$. It is well known that for any finite dimensional complex simple Lie algebra, all closed partitions of the root system are Weyl group conjugate to the standard partition. However, this is not the case for affine Lie algebras. The classification of closed subsets of the root system for affine Lie algebras was obtained by Jakobsen and Kac [18,19], and independently by Futorny [11,12]. In fact, for affine Lie algebras there exists a finite number (≥ 2) of inequivalent Weyl orbits of closed partitions. Corresponding to each such non-standard partitions we have non-standard Borel subalgebras from which we can induce other non-standard Verma-type modules and these typically contain both finite and infinite dimensional weight spaces. The imaginary Verma module [13] is a non-standard Verma-type module associated with the simplest non-standard partition of the root system Δ which is the focus of our study in this paper.

For generic q, the quantum affine algebra $U_q(\hat{\mathfrak{g}})$ is the q-deformations of the universal enveloping algebras of $\hat{\mathfrak{g}}$ [8,16]. It is known [25] that integrable highest weight modules of $\hat{\mathfrak{g}}$ can be deformed to those over $U_q(\hat{\mathfrak{g}})$ in such a way that the dimensions of the weight spaces are invariant under the deformation. Following the framework of [25] and [21], *quantum imaginary Verma modules* for $U_q(\hat{\mathfrak{g}})$ were constructed in [6,9] and it was shown that these modules are deformations of those over the universal enveloping algebra $U(\hat{\mathfrak{g}})$ in such a way that the weight multiplicities, both finite and infinite-dimensional, are preserved.

Kashiwara [22,23] from algebraic viewpoint and Lusztig [26] from geometric viewpoint introduced global crystal base (equivalently, canonical base) for standard Verma modules $V_q(\lambda)$ and integrable highest weight modules $L_q(\lambda)$. The crystal base [22,23] can be thought of as the $q = 0$ limit of the global crystal base or canonical base. An important ingredient in the construction of crystal base by Kashiwara in [23] is a subalgebra $\mathcal{B}_q$ which acts on the negative part of the quantum group by left multiplication. This subalgebra $\mathcal{B}_q$, which we call the Kashiwara algebra, played an important role in the definition of the Kashiwara operators which defines the crystal base.

In this paper we construct an analog of Kashiwara algebra $\mathcal{K}_q$ for the imaginary Verma module $M_q(\lambda)$ for the quantum affine algebra $U_q(\hat{\mathfrak{g}})$ by introducing certain Kashiwara-type operators. Then we prove that certain quotient $\mathcal{N}_q^-$ of $U_q(\hat{\mathfrak{g}})$ is a simple $\mathcal{K}_q$-module. This generalizes the corresponding result in [4] for the quantum affine algebra $U_q(\widehat{sl(2)})$. However, it is worth pointing out that some of the arguments involving explicit calculations in [4] do not extend to this general case.

The paper is organized as follows. In Sections 2 and 3 we recall necessary definitions and some results that we need. In Section 4 we recall some facts about the imaginary