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Classifying forms of simple groups via their invariant polynomials

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ABSTRACT

Let G be a simple linear algebraic group over a field F, and V an absolutely irreducible representation of G. We show that under some mild hypotheses there exists an invariant homogeneous polynomial f for the action of G on V defined over F, such that twisted forms of f up to a scalar multiple classify twisted forms of G for which the representation V is defined over F. This result extends the classical case of a quadratic form q and its orthogonal group $\mathrm{O}(q)$.

1. Introduction

It is a classical result that the isomorphism class of a nondegenerate quadratic form q is determined up to scalar multiple by the isomorphism class of the corresponding orthogonal group $\mathrm{O}(q)$ and this correspondence doesn't depend on the embedding of $\mathrm{O}(q)$ into the vector space underlying q.[1] In this paper we prove an extension of this result for more general algebraic groups. Explicitly, let G be a simple linear algebraic group over a field F and let V be an absolutely irreducible representation of G. A result of Garibaldi–Guralnick [6] states that in "most cases" there exists a homogeneous polynomial f on V

[1] Although the result is also true for fields of characteristic 2 as follows from our results, this doesn't seem to be as well known, see the discussion in [10] and [11].

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which is invariant under the action of G and such that the identity component of the scheme-theoretic stabilizer of f is G. We show that under these hypotheses, and more generally if the similarity group of f has identity component $\mathbb{G}_m \cdot G$, that one can construct a *maximally stable* polynomial which additionally has similarity group "as large as possible".

For such an invariant polynomial we show that, in many cases, similarity classes of f classify twisted forms of G for which V is defined over F up to isomorphism. In particular, if S denotes the similarity group of f (see below for definitions), and S is smooth, then the Galois cohomology set $\mathrm{H}^1(F, S)$ classifies twisted forms of f up to similarity, and the set $\mathrm{H}^1(F, \mathrm{Aut}(G))$ classifies twisted forms of G up to isomorphism. We define a map,

$$\phi : \mathrm{H}^1(F, S) \to \mathrm{H}^1\big(F, \mathrm{Aut}(G)\big), \tag{1}$$

which when the stabilizer of f is smooth can be interpreted as being given by mapping a twisted form of f to the identity component of its stabilizer.

Notice that in the case where $G = \mathrm{SO}(q)$ the classical result discussed above can be rephrased as stating that ϕ is injective.

This setup leads to some natural questions:

(1) What are the conditions on a twisted form f' of f to map to G under ϕ? That is to say, what are the fibers of ϕ?
(2) Which twisted forms of G appear as identity components of stabilizers of forms of f? In other words, what is the image of ϕ?

We answer these questions in detail in the sections below culminating in the following theorem.

Theorem 1.1. *Let f be a maximally stable form for a simple group G and put S for the similarity group of f. The map $\phi_E : \mathrm{H}^1(E, S) \to \mathrm{H}^1(E, \mathrm{Aut}(G))$ is an isomorphism for all field extensions E of F if and only if the highest weight λ of V is in the root lattice of G and is fixed by every automorphism of the Dynkin diagram of G.*

Proof. The theorem follows immediately from Corollaries 5.3 and 4.10. □

Given an adjoint group G then with some restrictions on the characteristic of F a representation for which the theorem applies always exists:

Theorem 1.2. *Let G be an adjoint simple linear algebraic group over a field F of characteristic such that the adjoint representation of G is irreducible,*[2] *and not 2 or 3 then*

[2] By [7, Table 1] this is not the case only if F has characteristic 2 and G has type B_n, C_n, D_n, E_7 or F_4, or F has characteristic 3 and G has type E_6 or G_2 or if G has type A_n and F has characteristic p with $p \mid n + 1$.