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Arithmetical rank of Gorenstein squarefree monomial ideals of height three

Kyouko Kimura [a,*], Naoki Terai [b]

[a] *Department of Mathematics, Graduate School of Science, Shizuoka University, 836 Ohya, Suruga-ku, Shizuoka 422-8529, Japan*
[b] *Department of Mathematics, Faculty of Culture and Education, Saga University, Saga 840-8502, Japan*

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ABSTRACT

We prove that a squarefree monomial ideal of height 3 whose quotient ring is Gorenstein is a set-theoretic complete intersection.

1. Introduction

Let S be a polynomial ring over a field K and I a squarefree monomial ideal of S. The *arithmetical rank* of I is defined as the minimum number u of elements $q_1, \dots, q_u \in I$ such that

$$\sqrt{(q_1, \dots, q_u)} = I,$$

* Corresponding author.
E-mail addresses: skkimur@ipc.shizuoka.ac.jp (K. Kimura), terai@cc.saga-u.ac.jp (N. Terai).

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where for an ideal J, $\sqrt{J}$ denotes the radical ideal of J. When this is the case, we say that $q_1, \dots, q_u$ *generate* I *up to radical*. Lyubeznik [15] proved that the projective dimension of S/I over S, denoted by $\mathrm{pd}_S\, S/I$ gives a lower bound for the arithmetical rank of a squarefree monomial ideal I. Thus we have the following inequalities:

$$\operatorname{height} I \leq \mathrm{pd}_S\, S/I \leq \operatorname{ara} I. \tag{1.1}$$

An ideal I is said to be a *set-theoretic complete intersection* if $\operatorname{height} I = \operatorname{ara} I$ holds. Note that a set-theoretic complete intersection ideal I is a Cohen–Macaulay ideal, that is, S/I is Cohen–Macaulay. From (1.1), the following natural question arises:

Question 1.1. Let I be a squarefree monomial ideal of S. When does $\operatorname{ara} I = \mathrm{pd}_S\, S/I$ hold? In particular, which ideal I is a set-theoretic complete intersection?

Many authors considered this question and found some classes of ideals with the equality; see, for example, [1,3–6,8–14,16–19]. In particular, in [9], the first author proved that a Cohen–Macaulay squarefree monomial ideal of height 2 is a set-theoretic complete intersection. However there is a counter-example found by Yan [20] among Cohen–Macaulay squarefree monomial ideals of height 3, though its Cohen–Macaulayness depends on the characteristic of the base field K. Then how about the Gorenstein squarefree monomial ideal of height 3? We give a positive answer to this question. Precisely the following theorem is the main result of the paper.

Theorem 1.2. *Let I be a Gorenstein squarefree monomial ideal of height* 3. *Then* $\operatorname{ara} I = 3$.

In particular, I is a set-theoretic complete intersection.

To prove Theorem 1.2, we first determine Gorenstein squarefree monomial ideals of height 3 by using the result by Bruns and Herzog [7]. Then it turns out that we only need to consider the ideals I_r (see Section 2).

Therefore if we construct 3 elements which generate I_r up to radical, then we have $\operatorname{ara} I_r \leq 3$ and conclude that $\operatorname{ara} I_r = 3$ and I_r is a set-theoretic complete intersection. Thus Theorem 1.2 is proved.

At the end of Section 2, we prove that I_r can be written as $J_r + x_0 J'_{r-1}$, where J_r is a squarefree monomial ideal (Lemma 2.5). In Section 3, we construct 2 elements which generate $x_0 J_r$ up to radical (Proposition 3.1). Our construction here is justified by the result by Schmitt and Vogel [19, Lemma p. 249]; see Lemma 3.2. Finally in Section 4, we construct 3 elements which generate I_r up to radical by using two elements constructed in Section 3. The key idea of the construction is a linear algebraic consideration which is explored by Barile [2] and used in several papers, for example, [5,6,9]; see Lemma 4.1.